Aut-invariant quasimorphisms on free products and generalisations

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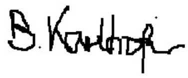
Declaration

I declare that this thesis has been composed entirely by myself and that the work contained within is my own. No portion of the work contained in this document has been submitted in support of an application for a degree or qualification at this or any other university or other institution of learning. All verbatim extracts have been distinguished by quotation marks and all sources of information have been specifically acknowledged.

Signed:



Abstract

This thesis constructs unbounded quasimorphisms that are invariant under all automorphisms on free products of groups and on graph products of finitely generated abelian groups. Moreover, we prove that the space of such quasimorphisms is infinite dimensional for these groups. Our constructions apply to many classes of right angled Artin and right angled Coxeter groups. We discuss various geometrically arising families of graphs as examples and deduce the non-triviality of an invariant analogue of stable commutator length recently introduced by Kawasaki and Kimura for these groups.

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Contents

1 Introduction

The study of quasimorphisms on a given group G is an important branch of geometric group theory with quasimorphisms carrying deep information of the underlying structure of the group G. A quasimorphism is a map $q\colon G \to \mathbb{R}$ satisfying $|q(gh) - q(g) - q(h)| \leq D$ for all $g, h \in G$ where $D \geq 0$. It is called homogeneous if $q(g^n) = n \cdot q(g)$ for all $g \in G$. Quasimorphisms have a very wide range of applications. They can be used to study the growth of normal generating sets of G, the structure of diffeomorphism groups, the dynamics in symplectic geometry and have relationships with algebraic invariants like the stable commutator length and the bounded cohomology of G. Quasimorphisms can even be used as a way of constructing the real numbers from the integers [A'Ca03].

For free groups F_n the so-called counting quasimorphisms originating from the work of Brooks in [Bro81] yield a wide variety of examples. His ideas have been developed further by Calegari and Fujiwara who constructed unbounded quasimorphisms on non-elementary hyperbolic groups [CaFu10]. Another rich source of examples are diffeomorphism groups and groups of diffeomorphisms that preserve additional structures like symplectomorphisms or Hamiltonian diffeomorphisms. For diffeomorphism groups of surfaces many important constructions are given in [GaGh04]. Quasimorphisms on mapping class groups of surfaces are for example discussed in [Kot04]. Numerous applications of quasimorphisms in symplectic geometry originate from work of Entov and Polterovich [EnPo03] together with Py [EPP12]. Brandenbursky proved many results involving quasimorphisms on groups of Hamiltonian diffeomorphisms [Bra15] and applications of quasimorphisms to dynamics are flourishing [FOOO19]. A fundamental paper on the geometry of quasimorphisms and central extensions is [BaGh92]. As a whole, finding quasimorphisms and studying stable commutator length has been an intensively researched area and remains a very active topic of research today even for the case of surface diffeomorphism groups [BHW21].

From the viewpoint of classical group theory many questions about the properties of a given group G concern the growth behaviour of normal generating sets of G. In the case of finite groups these questions have been extensively studied [LiPy97] and strong results for the case that G is simple had already been established in the 1970s and before, for example by Brenner et al. in a series of nine papers under the title "Covering theorems of finite nonabelian simple groups". The Ore conjecture for which the last remaining cases were famously established in [LOST10] states that in fact every element in the commutator subgroup of a finite simple group can be expressed as a single elementary commutator. From this algebraic perspective quasimorphisms can be perceived as a simple tool to study the growth behaviour of normal generating sets for infinite groups G.

Indeed, the word norm of any normal generating set S of a group G defines a so-called conjugation invariant norm. That is, a map $\|.\|\colon G \to \mathbb{R}_{\geq 0}$ that vanishes only on the neutral element, is invariant under inverting elements and conjugation and satisfies the triangle inequality. The existence of an unbounded quasimorphism that is bounded on the generating set S implies that this word norm becomes arbitrarily large on powers of some elements in G. In particular, it implies that the word norm defined by S has infinite diameter on G.

In the context of geometric group theory conjugation invariant norms often arise from geometrically meaningful generating sets S. For example, for diffeomorphism groups the fragmentation norm is a conjugation invariant norm measuring how a given quasimorphism on a manifold can be decomposed into quasimorphisms that are each supported on a ball in that manifold. The so-called Hofer norm is a conjugation invariant norm measuring decomposability for Hamiltonian diffeomorphisms. In [KKMT21] the author proved together with Kedra, Marcinkowski and Trost, that the generating set of closed geodesics defines unbounded conjugation invariant norms on hyperbolic manifolds that admit a certain geodesic symmetry. However, the existence of an unbounded conjugation invariant norm on a group G does not imply the existence of an unbounded quasimorphism on G as we will see in Section 2.1. An interesting example, the commutator subgroup of the infinite braid group, is discussed in [BrKe15].

In the case of the commutator subgroup of a group the notion of a conjugation invariant word norm generated by the elementary commutators of G leads to so-called commutator length $\mathrm{cl}\colon [G,G] \to \mathbb{R}_{\geq 0}$. In fact, historically, quasimorphisms were used to provide the first examples of perfect groups in which the commutator length is not a bounded function; that is, perfect groups which are not uniformly perfect. By taking the limit of the powers of a given element one obtains the stable commutator length $\mathrm{scl}(g) = \lim \frac{cl(g^n)}{n}$ for $g \in G$. A very thorough analysis of this invariant was given by Calegari in [Cal09]. Quasimorphisms share a deep relationship with the stable commutator length via Bavard's duality theorem [Bav91]. This theorem states that $\mathrm{scl}(g) = \frac{1}{2} \sup \frac{|\phi(g)|}{D(\phi)}$ for all $g \in G$ where the supremum is taken over the space of homogeneous quasimorphisms modulo the space of homomorphisms on G.

The study of quasimorphisms is instrumental for the theory of bounded cohomology on G since the second bounded cohomology group of G can be identified with the space of homogeneous quasimorphisms modulo the space of homomorphisms on G. As such they provide vital information for understanding the geometry of a group via machinery from algebraic topology. In fact, the information that the bounded cohomology of a topological space with real coefficients contains is completely encoded in the bounded cohomology of its fundamental group. However, computations in bounded cohomology are in general more difficult than in standard group cohomology. Especially for the case of the free group of rank two there have been many recent attempts to fully determine the cup product structure on $H^2_b(F_2, \mathbb{R})$ for various classes of quasimorphisms [Heu17], [BuMo18], [Fou20].

In this thesis we construct unbounded Aut-invariant quasimorphisms that are invariant under the action of automorphisms on free products of groups and graph products of finitely generated abelian groups. In Section 2 we introduce basic terminology, discuss initial examples and give general constructions how to construct new quasimorphisms from old ones. After recalling the structure of the automorphism group of a free product of groups in Section 3 we proceed to give our first constructions of Aut-invariant quasimorphisms on free products of two factors in Section 4. To achieve this we associate tuples of natural numbers that we call codes to each element in a free product $G = A * B$. Inspired by Brooks's counting quasimorphisms on free groups we then count occurences of these codes rather than occurences of words in the free product. We verify that this indeed yields quasimorphisms on G and call them code quasimorphisms. Using an explicit description of the automorphism group $\mathrm{Aut}(G)$ found in [Gil87] we see in Proposition 4.11 that our code quasimorphisms are unbounded and invariant with respect to all automorphisms of G if A and B are not infinite cyclic.

If one of the factors of $G = A * B$ is infinite cyclic, our code quasi-morphisms are not necessarily invariant under a specific class of automorphisms of G which is called the class of transvections. So, in Section 5 we slightly adjust the way we count codes for infinite cyclic factors and call the resulting maps weighted code quasimorphisms. We show in Proposition 5.7 that these are unbounded and invariant with respect to all automorphisms of G.

These two propositions together with an independent result for the free group on two generators from [BrMa19, Theorem 2] comprise the following result in Section 6.

Theorem (6.1)**.** Let $G = A * B$ be the free product of two non-trivial freely indecomposable groups A and B. Assume G is not the infinite dihedral group. Then G admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms, all of which vanish on single letters.

The infinite dihedral group does not admit an unbounded quasimorphism since all its elements are conjugate to their inverses. As a corollary of our construction we can immediately deduce the existence of stably unbounded Aut-invariant norms on free products of two factors.

Corollary (6.2)**.** Let $G = A * B$ be the free product of two non-trivial freely indecomposable groups and assume G is not the infinite dihedral group. Then there exists a stably unbounded Aut-invariant norm on G.

In Section 7 we proceed to generalise these results to free products of more than two factors and obtain the following theorem.

Theorem (7.3)**.** Let $G = G_1 * \cdots * G_k$ be a free product of freely indecomposable groups G_i where $k \geq 2$. Assume that at most two factors are infinite cyclic and there exists $j \in \{1, \ldots, k\}$ such that $G_j \ncong \mathbb{Z}/2$. Then G admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms, all of which vanish on single letters.

In Section 8 we proceed by considering graph products and their automorphism groups. Graph products generalise the construction of a free product by introducing commutator relations according to the edges of an underlying graph Γ. In Section 9 we define an equivalence relation $\sim_\tau$ on the vertex set of Γ which encodes the existence of a special class of automorphisms of the graph product. The notion of lower cones from [Mar20] and an explicit description of the automorphism group of graph products of finitely generated abelian groups given in [CoGu09] enables us to construct unbounded quasimorphisms that are invariant under a finite index subgroup $\mathrm{Aut}^0(G) \leq \mathrm{Aut}(G)$, where G is a graph product of finitely generated abelian groups. An averaging procedure will then produce $\mathrm{Aut}(G)$-invariant quasimorphisms that are still unbounded. For the case of right angled Artin groups we prove the existence of unbounded Aut-invariant quasimorphisms for many classes of graphs in Proposition 9.23 from which we obtain the following theorem as a special case.

Theorem (9.24)**.** Let $\Gamma = (V, E)$ be a finite graph with $|V| \geq 2$ and such that no two distinct vertices $x, y \in V$ satisfy $lk(v) \subset st(v)$. Then the right angled Artin group R_Γ admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms.

Our construction yields unbounded Aut-invariant quasimorphisms for a very large class of graph products of finite abelian groups in Theorem 9.27 answering questions from [Mar20] on their existence. From this we deduce the following theorem.

Theorem (9.30)**.** $\Gamma = (V, E)$ be a finite graph that is not complete and let W_Γ be a graph product of finite abelian groups on Γ. If there are no two vertices $v, w \in V$ such that $G_v = G_w = \mathbb{Z}/2$ and $lk(v) = lk(w)$, then W_Γ admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms.

As examples we discuss various families of graphs of geometric origin explicitly in Section 10. In Section 11 we turn to the so-called Aut-invariant stable commutator length that was recently introduced by Kawasaki and Kimura in [KaKi20] and was also shown to satisfy an invariant analogue of Bavard's duality theorem. We explicitly construct elements on which this invariant is non-trivial for free products as well as many graph products among which are the ones considered in both theorems above. For example, in the case of free products of two factors we conclude the following theorem.

Theorem (11.6)**.** Let $G = A * B$ be a free product of freely indecomposable groups and assume that G is not the infinite dihedral group. Then there always exist elements $g \in G$ with positive Aut-invariant stable commutator length $\mathrm{scl}_{\mathrm{Aut}}(g) > 0$.

Finally, there are two appendices to this thesis. In Appendix A we explain how one can already obtain unbounded Aut-invariant quasimorphisms on free products of two non-isomorphic finite groups purely from Brooks's quasimorphisms. In Appendix B we prove that the spaces of homogeneous Aut-invariant quasimorphisms on the braid group B_3, the special linear group $\mathrm{SL}(2, \mathbb{Z})$ and the projective special linear group $\mathrm{PSL}(2, \mathbb{Z})$ are isomorphic.

Note that large parts of my results on Aut-invariant quasimorphisms of free products of two factors are encapsulated in my research paper [Kar21(1)], whereas large parts of my results on graph products are consolidated in my research paper [Kar21(2)].

2 Preliminaries

Definition 2.1. Let G be a group. Denote by $\mathrm{Aut}(G)$ the group of all automorphisms of G. Furthermore, denote the normal subgroup of inner automorphisms by $\mathrm{Inn}(G)$ and the group of outer automorphisms by $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Definition 2.2. A subgroup $H \leq G$ is called *characteristic* if $\varphi(H) = H$ for all $\varphi \in \mathrm{Aut}(G)$.

Definition 2.3. A group homomorphism $f : G \to H$ is called *Aut-equivariant* if for all automorphisms $\varphi \in \mathrm{Aut}(G)$ there exists $\varphi_0 \in \mathrm{Aut}(H)$ such that $f \circ \varphi = \varphi_0 \circ f$.

Definition 2.4. Let G be a group. A map $\psi : G \to \mathbb{R}$ is called a *quasimorphism* if there exists a constant $D \geq 0$ such that

$$|\psi(g) + \psi(h) - \psi(gh)| \leq D \text{ for all } g, h \in G.$$

The *defect* of ψ is defined to be the smallest number $D(\psi)$ with the above property. A quasimorphism is *homogeneous* if it satisfies $\psi(g^n) = n\psi(g)$ for all $g \in G$ and all $n \in \mathbb{Z}$. Further, ψ is called Aut-invariant if $\psi(\varphi(g)) = \psi(g)$ for all $g \in G$, $\varphi \in \mathrm{Aut}(G)$.

It follows from the definition that sums and scalar multiples of (homogeneous) quasimorphisms are (homogeneous) quasimorphisms again. It is immediate that homogeneous quasimorphisms vanish on elements of finite order. Moreover, homogeneous quasimorphisms are constant on conjugacy classes since for a homogeneous quasimorphism $\psi : G \to \mathbb{R}$ and $g, h \in G$ we calculate

$$\begin{aligned} |\psi(g) - \psi(hgh^{-1})| &= \left| \lim_{n \in \mathbb{N}} \frac{\psi(g^n)}{n} - \lim_{n \in \mathbb{N}} \frac{\psi((hgh^{-1})^n)}{n} \right| = \lim_{n \in N} \frac{|\psi(g^n) - \psi(hg^n h^{-1})|}{n} \\ &\leq \lim_{n \in N} \frac{2D_\psi}{n}, \end{aligned}$$

which implies $\psi(g) = \psi(hgh^{-1})$.

Definition 2.5. Let $\psi : G \to \mathbb{R}$ be a quasimorphism. The *homogenisation* $\bar{\psi} : G \to \mathbb{R}$ of ψ is defined to be $\bar{\psi}(g) = \lim_{n \in \mathbb{N}} \frac{\psi(g^n)}{n}$ for all $g \in G$. To improve legibility we will denote the homogenisation of a quasimorphism with upper subscripts ψ^G by $\bar{\psi}^G$ instead of $\bar{\psi^G}$.

Lemma 2.6 ([Cal09, p.18])**.** *The homogenisation $\bar{\psi}$ of a quasimorphism $\psi : G \to \mathbb{R}$ is a homogeneous quasimorphism. Moreover, it satisfies $|\bar{\psi}(g) - \psi(g)| \leq D(\psi)$ for any $g \in G$.*

Definition 2.7. A function $\nu\colon G \to \mathbb{R}$ satisfying for all $g, h \in G$:

- $\nu(g) \geq 0$,
- $\nu(g) = 0$ if and only if $g = 1$,
- $\nu(gh) \leq \nu(g) + \nu(h)$,

is called a *norm* on G. If in addition for all $g \in G$ and $\varphi \in \mathrm{Aut}(G)$ it satisfies

- $\nu(\varphi(g)) = \nu(g)$,

then it is called *Aut-invariant*. The supremum $\nu(G) = \sup\{\nu(g) \mid g \in G\}$ is called the *diameter* of the norm ν. If $\nu(G) = \infty$, then ν is called *unbounded*. If there exists $g \in G$ such that $\lim_{n\to\infty} \frac{\nu(g^n)}{n} > 0$, then ν is called *stably unbounded*.

Example 2.8. Let G be a group together with a generating set S. The *word norm* generated by S is the norm on G defined by

$$\nu_S(g) = \min\{n \mid g = s_1 \cdots s_n \text{ where } n \in \mathbb{N} \text{ and } s_i \in S \text{ for all } i\}.$$

If we assume additionally that the set S is invariant under $\mathrm{Aut}(G)$ then ν_S is Aut-invariant.

Lemma 2.9. *Let $\psi\colon G \to R$ be an Aut-invariant quasimorphism with unbounded image, but bounded on a generating set S of G. Then there exists a stably unbounded Aut-invariant norm on G.*

Proof. By Lemma 2.6 we can assume that ψ is homogeneous. The word norm $\|.\|_{\bar{S}}$ on G associated to the generating set $\bar{S} = \{\varphi(s) \mid s \in S, \varphi \in \mathrm{Aut}(G)\}$ is clearly Aut-invariant. Let K be a positive bound for the absolute value of ψ on S. Write $g \in G$ as a product of these generators $g = \varphi_1(s_1) \cdots \varphi_n(s_n)$ for some $n \in \mathbb{N}$ where $s_i \in S$ and $\varphi_i \in \mathrm{Aut}(G)$ for all i. The calculation

$$\begin{aligned}|\psi(g)| = |\psi(\varphi_1(s_1) \cdots \varphi_n(s_n))| &\leq |\psi(\varphi_1(s_1))| + \cdots + |\psi(\varphi_n(s_n))| + (n-1)D(\psi)\\ &\leq n(K + D(\psi))\end{aligned}$$

shows that $\|g\|_{\bar{S}} \geq \frac{|\psi(g)|}{K+D(\psi)}$ for all $g \in G$. It follows that $\|g^k\|_{\bar{S}} \geq k \cdot \frac{|\psi(g)|}{K+D(\psi)}$ for all $k \in \mathbb{N}$, $g \in G$. Since ψ does not vanish everywhere, $\|.\|_{\bar{S}}$ is a stably unbounded Aut-invariant norm on G. ☐

2.1 Initial examples

Let us illustrate the question on the existence of Aut-invariant quasimorphisms by considering some initial examples. In particular, let us see that the converse of Lemma 2.9 above is not true and finding unbounded Aut-invariant quasi-morphisms is much more difficult than finding unbounded Aut-invariant norms in general.

Example 2.10. Let Σ_∞ be the infinite symmetric group of finitely supported bijections of the natural numbers. The cardinality of the support defines an Aut-invariant norm of infinite diameter on Σ_∞. However, any element $g \in \Sigma_\infty$ has finite order. Therefore, no Aut-invariant norm on Σ_∞ is stably unbounded and any homogeneous quasi-morphism vanishes on all of Σ_∞. Consequently, by Lemma 2.6 any quasimorphism on Σ_∞ is bounded.

Example 2.11. Let G be an abelian group. Then G does not admit an unbounded Aut-invariant quasimorphism. In fact, the inversion $\iota\colon G \to G$ defined by $\iota(g) = g^{-1}$ is an automorphism since G is abelian. Let $\psi\colon G \to \mathbb{R}$ be an Aut-invariant quasimorphism. By definition $\bar{\psi}$ is Aut-invariant as well and by Lemma 2.6 $\bar{\psi}$ is only a finite distance away from ψ. However, $\bar{\psi}(g) = \bar{\psi}(\iota(g)) = \bar{\psi}(g^{-1}) = -\bar{\psi}(g)$ for all $g \in G$. So $\bar{\psi}$ vanishes and ψ was bounded to begin with.

Example 2.12. Let $G = \mathbb{Z}^k$ for $k \geq 1$. For $k = 1$ the standard absolute value defines a stably unbounded Aut-invariant norm on G, whereas for $k \geq 2$ any Aut-invariant norm on G has finite diameter.

Example 2.13. Let $D_\infty = \mathbb{Z}/2*\mathbb{Z}/2$ be the infinite dihedral group. Then $(D_\infty)^k$ does not admit an unbounded Aut-invariant norm for any $k \geq 1$. Indeed, any element in D_∞ is the product of at most two conjugates of the standard generating set. Consequently, any norm on D_∞ that is invariant under conjugation is bounded by twice the maximal value on the standard generating set. Thus, for all $k \geq 1$ any Aut-invariant norm on $(D_\infty)^k$ is bounded as well and $(D_\infty)^k$ cannot admit an unbounded Aut-invariant quasimorphism by Lemma 2.9. The case of $Z \leq D_\infty$ also shows that quasimorphisms of finite index subgroups can in general not be extended to the full group.

Example 2.14. Let G be the fundamental group of the Klein bottle $G = \mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z}$. Let a and b be generators of the two infinite cyclic factors of G in its above presentation. Consider the Aut-invariant word norm ν_S generated by $S = \{\varphi(a^{\pm 1}), \varphi(b^{\pm 1}) \mid \varphi \in \mathrm{Aut}(G)\}$. To see that this norm is unbounded on G we first note that commutator subgroup of G is a characteristic subgroup and $G/[G,G] = \mathbb{Z}/2 \times \mathbb{Z}$, where $\mathbb{Z}/2$ is a characteristic subgroup. Consequently, the projection map $p\colon G \to \mathbb{Z}$ sending $p(a) = p(b) = 1$ is Aut-equivariant and maps the set S to the Aut-invariant set $\{\pm 1\}$ in $\mathbb{Z}$, which generates a stably unbounded Aut-invariant norm on $\mathbb{Z}$. Therefore, the word norm ν_S generated by S on G is stably unbounded as well.

However, there is no unbounded Aut-invariant quasimorphism on G. If ψ was such a quasimorphism, it could be chosen to be homogeneous by Lemma 2.6. Let φ be the automorphism inverting the generators a and b. Then φ inverts the center $Z(G) = 2\mathbb{Z}$ as well. Hence, ψ vanishes on $Z(G)$. Similarly, every element of $S = \{(ab)^n, (ba)^n, (ab)^n a, (ba)^n b \mid n \in \mathbb{N}\}$ belongs to the same Aut-orbit that its inverse belongs to. So ψ vanishes on S as well. However, every element $g \in G$ can be written as a product $= zs$ where $z \in Z(G)$ and $s \in S$. Therefore, ψ is bounded on all of G.

Example 2.15. Let B_3 be the braid group on three strands. Then the homomorphism $B_3 \to \mathbb{Z}$ sending the standard generators to 1 is Aut-equivariant. Since the absolute value is an unbounded Aut-invariant norm on $\mathbb{Z}$, this shows that the Aut-orbits of the standard generators and their inverses generate an unbounded Aut-invariant norm on B_3. However, since the non-trivial outer automorphism of B_3 inverts all standard generators, this map cannot be modified to construct an unbounded quasimorphism on B_3. We will see in Example 6.8 that B_3 does admit unbounded Aut-invariant quasimorphisms and we will further discuss the space of homogeneous Aut-invariant quasimorphisms on B_3 in Appendix B.

2.2 Quasimorphisms on free products

Let I be an indexing set and let $G = *_{i\in I} G_i$ be a free product of a family of groups $\{G_i\}_{i\in I}$. Via the canonical inclusion the factor G_i is a subgroup of G for each $i \in I$. An element of G belonging to one of the factors is called a *letter* of G. A *word* in G is a product of letters in

G. For any two letters belonging to the same factor in G their product in G can be replaced by the letter that represents their product in that factor. Moreover, any letters that are the identity in the factor they belong to can be omitted inside any word without changing the element that word represents in G.

A word is called *reduced* if no two consecutive letters belong to the same factor and no letters appear that represent the identity. Recall that every element $g \in G$ has a unique presentation as a reduced word. Finally, a word w is called *cyclically reduced* if all cyclic permutations if its letters are reduced words. This is equivalent to w being reduced such that its first and last letter belong to different factors.

Recall, that G has trivial center whenever there are at least two non-trivial free factors. In fact, no two letters of different groups commute by the uniqueness of reduced presentations. Moreover, if there was a non-trivial central element in G, we could choose a reduced representative w of a central element of minimal length $\ell(w) \geq 2$. Then the first and last letter of w belong to different factors since $\ell(w)$ was not minimal to begin with otherwise. Let x be the first letter of w. Then the reduced expressions of $x^{-1}w$ and wx^{-1} are different, which contradicts the assumption of w representing a central element.

Definition 2.16. A group G is called *freely indecomposable* if G is non-trivial and not isomorphic to a free product of the form $G_1 * G_2$ where G_1, G_2 are non-trivial groups.

For example, every finite group is freely indecomposable since any free product of non-trivial groups contains elements of infinite order. Every group with non-trivial center is freely indecomposable since any free product of non-trivial groups has trivial center.

Lemma 2.17. *Let I be a set of cardinality at least two. Let G_i be a non-trivial group for all i and $G = *_{i\in I} G_i$ be their free product. Let $\theta\colon G \to \mathbb{R}$ be a map whose absolute value is bounded on all letters of G by a constant $B \geq 0$. Assume that there exists a constant $D \geq 0$ such that*

$$|\theta(w_1 w_2) - \theta(w_1) - \theta(w_2)| \leq D$$

holds for all reduced words w_1, w_2 for which their product $w_1 w_2$ is a reduced word. Then the map $f\colon G \to \mathbb{R}$ defined by $f(w) = \theta(w) - \theta(w^{-1})$ defines a quasimorphism of defect at most $12D + 6B$, which is bounded on all letters by $2B$.

Proof. Any element in G can be represented by a reduced word. So let w_1, w_2 be reduced words. The word given by their product $w_1 \cdot w_2$ is reduced if and only if the last letter from w_1 belongs to a factor different from the one that the first letter of w_2 belongs to. Indeed, otherwise those two letters could be multiplied in their common factor and replaced by their product to shorten the number of letters appearing in the expression.

In order to bring $w_1 \cdot w_2$ to its reduced form we first perform all cancellations which form a word we call c. After all cancellations have taken place the final potential reduction is to possibly replace a non-trivial product of two letters b and d belonging to the same factor by a non-trivial letter x representing their product in that factor. Therefore, we have two cases.

- The reduced presentations of w_1 and w_2 are given by $w_1 = ac$, $w_2 = c^{-1}e$ and ae is the reduced presentation for $w_1 \cdot w_2$.
- The reduced presentations of w_1 and w_2 are given by $w_1 = abc$, $w_2 = c^{-1}de$, where b and d are letters belonging to the same factor. The reduced presentation of $w_1 \cdot w_2$ is given by axe, where $x = bd$ is the letter representing the non-trivial product of b and d.

We calculate for the second case that

$$\begin{aligned}
|f(w_1w_2)-&f(w_1)-f(w_2)| = |f(axe)-f(abc)-f(c^{-1}de)| \\
&= \left|\theta(axe)-\theta(e^{-1}x^{-1}a^{-1})-\theta(abc)+\theta(c^{-1}b^{-1}a^{-1})-\theta(c^{-1}de)+\theta(e^{-1}d^{-1}c)\right| \\
&\leq |\theta(a)+\theta(x)+\theta(e)-\theta(e^{-1})-\theta(x^{-1})-\theta(a^{-1})-\theta(a)-\theta(b)-\theta(c)+\theta(c^{-1}) \\
&\quad +\theta(b^{-1})+\theta(a^{-1})-\theta(c^{-1})-\theta(d)-\theta(e)+\theta(e^{-1})+\theta(d^{-1})+\theta(c)|+12D \\
&= |\theta(x)-\theta(x^{-1})-\theta(b)+\theta(b^{-1})-\theta(d)+\theta(d^{-1})|+12D \\
&\leq 6B+12D.
\end{aligned}$$

The first case follows analogously. Since w_1, w_2 were arbitrary reduced words and every element of G can be written in its reduced form, f is a quasimorphism of defect at most $6B+12D$. Since θ is bounded on all letters by B, so is f by $2B$. □

2.3 Brooks quasimorphisms

Let $G = G_1 * \cdots * G_k$ be a free product and let w be a reduced word in G. Let

$$c_w(g) = \text{number of disjoint occurrences of } w \text{ in the reduced representative of } g.$$

Clearly, c_w satisfies the assumption of Lemma 2.17 and we call $f_w^{Br} = c_w - c_{w^{-1}}$ a *Brooks quasimorphism*. However, note that Brooks originally introduced his quasimorphisms for free groups in [Bro81] using counting functions C_w counting all occurrences of w in a reduced representative instead of just counting disjoint occurrences. We will only consider his quasimorphisms of the form f_w^{Br} throughout this thesis.

Let w be a non-trivial word in a free group F. If w is a letter, then there exists a homomorphism $F \to \mathbb{Z}$ such that $f(w) \neq 0$. A conjugate w_0 of w is cyclically reduced and we assume that w_0 has length ≥ 2. Then $f_{w_0}^{Br}(w_0^n) = n$ since no element in a free group is conjugate to its inverse. So the homomogenisation of $f_{w_0}^{Br}$ satisfies $\bar{f}_{w_0}^{Br}(w) = \bar{f}_{w_0}^{Br}(w_0) \neq 0$. Thus, Brooks's quasimorphisms show that in a free group every non-trivial element can be detected by a quasimorphism. This is very particular for free groups. The following remark shows that does not even remain true in more general free products like the hyperbolic group $\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/3 * \mathbb{Z}/2$.

Remark 2.18. In general, there is no reason to assume that every element of infinite order in a free product $A * B$ of freely indecomposable groups is detected by a homogeneous quasimorphism. For example, if $b \in B$ has order two and $a \in A$ is any non-trivial element, then the word $w = aba^{-1}b$ has infinite order. However, w is conjugate to its inverse $w^{-1} = baba^{-1}$ and therefore any homogeneous quasimorphism vanishes on w.

2.4 New quasimorphisms from old ones

Lemma 2.19. *Let $\psi\colon G \to \mathbb{R}$ be a quasimorphism. Let $\{\varphi_i\}_{i\in I}$ be a set of representatives for the elements of* $\mathrm{Out}(G)$*. If ψ is invariant under φ_i for all i, then its homogenisation $\bar{\psi}\colon G \to \mathbb{R}$ is invariant under all automorphisms of G.*

Proof. The homogenisation $\bar{\psi}$ is constant on conjugacy classes [Cal09, p.19]. By definition $\bar{\psi}$ is also invariant under the collection $\{\varphi_i\}_{i\in I}$, since ψ is. The result follows since any element $\varphi \in \mathrm{Aut}(G)$ can be written as the composition of some φ_j with a conjugation. □

Lemma 2.20. *Let $H \leq G$ be a subgroup of finite index $k \in \mathbb{N}$ and $I = \{g_1, \ldots, g_k\}$ be a set of right coset representatives. So $G = \bigcup_{i=1}^{n} Hg_i$ where the union is disjoint. Let $g \in G$ be arbitrary. For each $i \in \{1, \ldots, k\}$ we can write $g_i g = h_i g_{j_i}$ uniquely where $j_i \in \{1, \ldots, k\}$ and $h_i \in H$. Define*

$$J_g = \{g_{j_i} \in I \mid \exists i \in \{1, \ldots, k\} \text{ such that } g_i g = h_i g_{j_i} \text{ where } h_i \in H\}.$$

Then $J_g = I$.

Proof. Let $g \in G$. Since I is a system of coset representatives, the product of g_i with g can indeed be written for all $i \in \{1, \ldots, k\}$ as $g_i g = h_i g_{j_i}$ where $h_i \in H$ and $j_i \in \{1, \ldots, k\}$. Then

$$G = Gg = \bigcup_{i=1}^{n} Hg_i g = \bigcup_{i=1}^{n} Hh_i g_{j_i} = \bigcup_{i=1}^{n} Hg_{j_i}$$

where the union is disjoint. Therefore, the set J_g consisting of all g_{j_i} appearing in this disjoint union needs to contain all coset representatives. That is $J_g = I$ independently of the choice of $g \in G$. □

Lemma 2.21. *Let $\psi \colon G \to \mathbb{R}$ be a quasimorphism invariant under a subgroup $H \leq \mathrm{Aut}(G)$ of index $k \in \mathbb{N}$. Let $\{f_1, \ldots, f_k\}$ be a set of right coset representatives. Let $\widehat{\psi} \colon G \to \mathbb{R}$ be defined by $\widehat{\psi}(g) = \sum_{i=1}^{k} \psi(f_i(g))$ for all $g \in G$. Then $\widehat{\psi}$ is an Aut-invariant quasimorphism on G.*

Proof. Clearly, $\widehat{\psi}$ is a quasimorphism as a finite sum of quasimorphisms. Let $\theta \in \mathrm{Aut}(G)$ and $g \in G$. For all i we can uniquely write $f_i \circ \theta = h_i \circ f_{j_i}$ where $j_i \in \{1, \ldots, k\}$ and $h_i \in H$ since $\{f_1, \ldots, f_k\}$ is a set of right coset representatives of H in $\mathrm{Aut}(G)$. We calculate

$$\begin{aligned} \widehat{\psi}(\theta(g)) &= \sum_{i=1}^{k} \psi(f_i(\theta(g))) = \sum_{i=1}^{k} \psi((f_i \circ \theta)(g)) = \sum_{i=1}^{k} \psi((h_i \circ f_{j_i})(g)) \\ &\overset{\text{H-invariance of } \psi}{=} \sum_{i=1}^{k} \psi(f_{j_i}(g)) \overset{\text{Lemma 2.20}}{=} \sum_{i=1}^{k} \psi(f_i(g)) = \widehat{\psi}(g). \end{aligned}$$

The penultimate equality follows from the fact that due to Lemma 2.20 the set of all f_{j_i} appearing in $\sum_{i=1}^{k} \psi(f_{j_i}(g))$ agrees with the set of all f_i appearing in $\sum_{i=1}^{k} \psi(f_i(g))$. □

Remark 2.22. Note, that the Aut-invariant quasimorphisms constructed from finite index subgroups in Lemma 2.21 are not necessarily unbounded. For example, this fails for the integers $\mathbb{Z}$ together with the identity map. We will in the following always verify unboundedness of quasimorphisms constructed in this way separately.

Lemma 2.23. *Let G be a group and $H \leq G$ be a characteristic subgroup with quotient projection $p \colon G \to G/H$. Then for any unbounded Aut-invariant quasimorphism $\psi \colon G/H \to \mathbb{R}$ the composition $\psi \circ p \colon G \to \mathbb{R}$ is an unbounded Aut-invariant quasimorphism on G. Moreover, linearly independent quasimorphisms on G/H give rise to linearly independent quasimorphisms on G.*

Proof. Clearly, $\psi \circ p$ is a quasimorphism. The Aut-invariance of $\psi \circ p$ on G follows from the Aut-invariance of ψ on G/H together with the fact that H is characteristic. Finally, the statement about linear independence follows from the surjectivity of the projection to the quotient. □

3 Automorphism group of free products

Let $G = G_1 * \cdots * G_k$ be a free product of freely indecomposable groups. Then following the exposition in [Gil87, p.116] based on results in [FoRa40] and [FoRa41] there are the following classes of automorphisms that generate $\mathrm{Aut}(G)$.

1. Applying the automorphism of any factor G_i to that factor defines an automorphism of G that is called a *factor automorphism*.
2. If two factors in G are isomorphic, interchanging those two factors defines an automorphism of G. Such an automorphism is called a *swap automorphism*.
3. Let $g \in G_i$ for some $i \in \{1, \dots k\}$. Define the map $p_g \colon G \to G$ to be conjugation by g on the letters of G_j for some $j \neq i$ and to be the identity on all other letters. This definition gives rise to an automorphism of G which is called a *partial conjugation*.

For each infinite cyclic factor G_i in G there are the following additional automorphisms:

4. Let s be a generator of G_i and let $a \in G_j$ be a letter where $j \neq i$. Then a *transvection* is the automorphism of G defined to be the identity on all letters belonging to factors that are not G_i and sending $s \to as$ or $s \to sa$

It was established in [FoRa40] and [FoRa41] that these classes generate the automorphism group of a free product.

Theorem 3.1 (Fouxe-Rabinovitch). *Let $G = G_1 * \cdots * G_k$ be a free product of freely indecomposable groups. Then $\mathrm{Aut}(G)$ is generated by the classes of swap automorphisms, factor automorphisms, transvections and partial conjugations.*

Definition 3.2. Denote by $\mathrm{Aut}^0(G)$ the subgroup of a free product of freely indecomposable groups $G = G_1 * \cdots * G_k$ generated by automorphisms of type 2–4.

In a free product $G = G_1 * \cdots * G_k$ any two choices of swap automorphisms interchanging two factors G_i and G_j differ by a product of factor automorphisms of these factors. Since the number of permutations of a finite set is finite one immediately concludes the following lemma.

Lemma 3.3. *Let $G = G_1 * \cdots * G_k$ be a free product of freely indecomposable groups. Then $\mathrm{Aut}^0(G)$ has finite index in $\mathrm{Aut}(G)$ with a system of coset representatives given by a choice of permutations of the isomorphic free factors of G.* □

Following the above description of the group of automorphisms of a free product of two factors we obtain:

Lemma 3.4. *Let G_1, G_2 be freely indecomposable groups such that G_2 is not infinite cyclic. Then the outer automorphism group of their free product $\mathrm{Out}(G_1 * G_2)$ is generated by the images of $\mathrm{Aut}(G_1)$, $\mathrm{Aut}(G_2)$ in $\mathrm{Out}(G_1 * G_2)$ together with a swap automorphism if $G_1 \cong G_2$ and the transvections if $G_1 \cong \mathbb{Z}$.*

Proof. By the universal property of the free product any automorphism of $G_1 * G_2$ is uniquely determined by its image on single letters. Let $h \in G_1$ and denote conjugation by h^{-1} on all of G by c_h. Then

$$(c_h \circ p_h)(g) = \begin{cases} h^{-1}gh & \text{if } g \in G_1, \\ g & \text{if } g \in G_2. \end{cases}$$

Thus, p_h and the factor automorphism given by conjugation by h^{-1} on G_1 represent the same element in $\mathrm{Out}(G)$. Similarly, in $\mathrm{Out}(G)$ partial conjugations on G_1 by elements from G_2 represent the same elements that factor automorphisms from G_2 do. Finally, any two choices of swap automorphism differ by a product of factor automorphisms. □

4 Code quasimorphisms

Recall that a *tuple* always refers to a finite sequence and so all tuples are naturally ordered.

Definition 4.1. Let A and B be groups. Write a given element $g \in A * B$ in its reduced form. We assign two tuples of non-zero natural number that we will call *codes* as follows. Let $(a_1, \ldots, a_k)$ be the tuple of letters from A appearing in the reduced form of g. We call $(a_1, \ldots, a_k)$ the *A-tuple* of g. Then we count how often any one letter of $(a_1, \ldots, a_k)$ appears consecutively. This yields a tuple of positive numbers $A\text{-code}(g) = (n_1, n_2, \ldots, n_r)$ which we call the *A-code* of g. Similarly, we obtain the ***B*-tuple**, which is the tuple of letters from B appearing in the reduced form of g, and the ***B*-code** of g, denoted $B\text{-code}(g)$, by counting consecutive appearances of letters in the ***B*-tuple**.

Note that $A\text{-code}(g)$ and $B\text{-code}(g)$ might have very different length for elements $g \in A*B$ in general.

Example 4.2. Let $G = A * B$ where $A = \mathbb{Z}/5$ and B is any group. Let $a \in A, b \in B$ be non-trivial elements. Consider $g = a^2bababa^4baba$. The A-tuple of g is (a^2, a, a, a^4, a, a) and therefore $A\text{-code}(g) = (1, 2, 1, 2)$. However, the B-tuple of g is (b, b, b, b, b) and so $B\text{-code}(g) = (5)$.

Remark 4.3. The code of any element $g \in A * B$ is clearly invariant under all factor automorphisms.

The following lemma is immediate.

Lemma 4.4. *The* A-code *and* B-code *of* g^{-1} *are the reversed* A- *and* B-code *of* g *for any* $g \in A * B$. *That is, let* $A\text{-code}(g) = (n_1, \ldots, n_k)$ *and* $B\text{-code}(g) = (m_1, \ldots, m_\ell)$*, then* $A\text{-code}(g^{-1}) = (n_k, \ldots, n_1)$ *and* $B\text{-code}(g^{-1}) = (m_\ell, \ldots, m_1)$. □

In the spirit of Brooks counting quasimorphisms we will now define *code quasimorphisms*, which are counting the occurrences of a string of natural numbers in the A-code and B-code associated to an element in the free product $A * B$.

Definition 4.5 (Code quasimorphisms)**.** Let $k \geq 1$ and let $z = (n_1, \ldots, n_k)$ be a tuple of non-zero natural numbers $n_1, \ldots, n_k$ for some $k \in \mathbb{N}$. Let $C \in \{A, B\}$. Define $\theta_z^C \colon A * B \to \mathbb{Z}_{\geq 0}$ to count the maximal number of *disjoint* occurrences of z as a tuple of consecutive numbers in the C-code for all $g \in A * B$. Further, define the *code quasimorphism*

$$f_z^C \colon A * B \to \mathbb{Z} \quad \text{by} \quad f_z^C(g) = \theta_z^C(g) - \theta_z^C(g^{-1})$$

for all $g \in A * B$. Note that $\theta_z^C(g^{-1}) = \theta_{\bar{z}}^C(g)$ due to Lemma 4.4, where $\bar{z}$ denotes the reversed tuple $(n_k, \ldots, n_1)$. Consequently, $f_z^C(g)$ can also be written as $f_z^C(g) = \theta_z^C(g) - \theta_{\bar{z}}^C(g)$ for all $g \in G$.

Example 4.6. Let $G = \mathbb{Z}/5 * B$ and $g = a^2bababa^4baba$ for non-trivial $a \in A, b \in B$ as in Example 4.2. For $z = (1, 2)$ we calculate $\theta_z^A(g) = 2$ and $\theta_z^A(g^{-1}) = \theta_{\bar{z}}^A(g) = 1$ and so $f_z^A(g) = 2 - 1 = 1$.

Example 4.7. Let $G = \mathbb{Z}/5 * B$ and $g = a^4bababa^3bababa^3$ for non-trivial $a \in A, b \in B$. In this case $A\text{-code}(g) = (1, 2, 1, 2, 1)$. Then $\theta_z^A(g) = 1$ for $z = (1, 2, 1)$ since we only count disjoint occurrences. Similarly, $\theta_z^A(g^{-1}) = \theta_{\bar{z}}^A(g) = 1$ and so $f_z^A(g) = 0$.

Lemma 4.8. *Let A, B be non-trivial groups and let $C \in \{A, B\}$. For a non-empty tuple of non-zero natural numbers z the map $f_z^C : A * B \to \mathbb{Z}$ defines a quasimorphism that is bounded on letters and invariant with respect to all factor automorphisms. Moreover, $D(f_z^C) \leq 30$.*

Proof. We want to apply Lemma 2.17 to deduce that f_z^C is a quasimorphism. Clearly, for all letters $x \in A * B$ and all z we have $|\theta_z^C(x)| \leq 1$. Let w_1, w_2 be reduced words representing elements in $A * B$ such that their product w_1w_2 is reduced. That is, the last letter of w_1 and the first letter of w_2 belong to different factors. Without loss of generality we can assume $C = A$. Let $A\text{-code}(w_1) = (n_1, \dots, n_k)$ and $A\text{-code}(w_2) = (m_1, \dots, m_\ell)$. Let x be the last letter from A in w_1 and let y be the first letter from A in w_2. Then

$$A\text{-code}(w_1w_2) = \begin{cases} (n_1, \dots, n_k, m_1, \dots, m_\ell) & \text{if } x \neq y, \\ (n_1, \dots n_{k-1}, n_k + m_1, m_2, \dots, m_\ell) & \text{if } x = y. \end{cases}$$

If $x \neq y$, then $\theta_z^C(w_1w_2) \in \{\theta_z^C(w_1) + \theta_z^C(w_2), \theta_z^C(w_1) + \theta_z^C(w_2) + 1\}$ since at most one of the disjoint occurrences of z can involve numbers that do not lie completely in the A-code of either w_1 or w_2.

If $x = y$, then $\theta_z^C(w_1w_2) \geq \theta_z^C(w_1) + \theta_z^C(w_2) - 2$ since n_k and m_1 can each be contained in at most one occurrences of z in the A-code of w_1 and w_2. Moreover, if an occurrence of z in the A-code of w_1w_2 involves $n_k + m_1$, then all other occurrences are fully contained in either the A-code of w_1 or w_2. Thus, $\theta_z^C(w_1w_2) \leq \theta_z^C(w_1) + \theta_z^C(w_2) + 1$.

In both cases we conclude

$$|\theta_z^C(w_1w_2) - \theta_z^C(w_1) - \theta_z^C(w_2)| \leq 2,$$

and it follows from Lemma 2.17 that f_z^C is a quasimorphism of defect $D(f_z^C) \leq 30$.

Moreover, by Remark 4.3 the maps θ_z^C are invariant under all factor automorphisms of $A * B$. Consequently, $f_z^C = \theta_z^C - \theta_{\bar{z}}^C$ is invariant under factor automorphisms as well. □

Definition 4.9. A tuple of non-zero natural numbers $z = (n_1, \dots, n_k)$ is called *generic* if $\bar{z}$ does not occur as a tuple of k adjacent numbers in $z^2 = (n_1, \dots, n_k, n_1, \dots n_k)$.

Example 4.10. Let $z = (n_1, \dots, n_k)$. If $k \leq 2$, z is not generic. If $k \geq 3$ and the n_i are pairwise distinct, then z is generic. E.g. for $z = (1, 2, 3)$ we have $\bar{z} = (3, 2, 1)$ does not appear in $z^2 = (1, 2, 3, 1, 2, 3)$.

Proposition 4.11. *Let $A * B$ be a free product of two freely indecomposable groups A and B, neither of which is infinite cyclic. Then for any generic tuple of natural numbers z the following holds:*

1. *if $A \not\cong B$ and $C \in \{A, B\}$ is such that $C \not\cong \mathbb{Z}/2$, then the homogenisation $\bar{f}_z^C$ of the quasimorphism f_z^C is an unbounded Aut-invariant quasimorphism on $A * B$;*
2. *if $A \cong B \not\cong \mathbb{Z}/2$, then the sum $\bar{f}_z^A + \bar{f}_z^B$ is an unbounded Aut-invariant quasimorphism on $A * B$.*

In both cases the space of homogeneous Aut-invariant quasimorphisms on $A * B$ that vanish on letters has infinite dimension.

Proof. First, consider the case $A \not\cong B$. By assumption at least one of the factors is not isomorphic to $\mathbb{Z}/2$. Without loss of generality we assume $A \not\cong \mathbb{Z}/2$. Let z be generic. By Lemma 4.8 the map f_z^A defines a quasimorphism invariant under all factor automorphisms. According to Lemma 3.4 this means that f_z^A is invariant under a full set of representatives for $\mathrm{Out}(A * B)$. Therefore, the homogenisation $\bar{f}_z^A$ is invariant under all automorphisms of $A * B$ by Lemma 2.19. It remains to check that $\bar{f}_z^A$ is unbounded, which is equivalent to checking that f_z^A itself is unbounded by Lemma 2.6.

Since $A \not\cong \mathbb{Z}/2$, it satisfies $|A| \geq 3$ and we can choose two distinct non-trivial elements $a_1, a_2 \in A$. Furthermore, choose a non-trivial element $b \in B$. Let $z = (n_1, \ldots, n_k)$ and choose $m \in \mathbb{N}$ to be non-zero and distinct from all $n_i \in \mathbb{N}$. We set

$$w_0 = (a_1 b)^{n_1} (a_2 b)^{n_2} (a_1 b)^{n_3} (a_2 b)^{n_4} \ldots (a_s b)^{n_k},$$

where $s = 1$ if k is odd and $s = 2$ if k is even. Set

$$w = \begin{cases} w_0 & \text{if } k \text{ is even,} \\ w_0 (a_2 b)^m & \text{if } k \text{ is odd.} \end{cases}$$

The $A\text{-code}$ of w is given by

$$A\text{-code}(w) = \begin{cases} (n_1, \ldots, n_k) = z & \text{if } k \text{ is even,} \\ (n_1, \ldots, n_k, m) = (z, m) & \text{if } k \text{ is odd.} \end{cases}$$

Since w starts and ends with letters from different groups, the reduced expression of w^ℓ is the ℓ-fold product of the word w for all $\ell \in \mathbb{N}$. Moreover, because the first letter from A in w is a_1 and the last letter from A is a_2, the $A\text{-code}$ of w^ℓ is

$$A\text{-code}(w^\ell) = \begin{cases} (z, z, \ldots, z) & \text{if } k \text{ is even,} \\ (z, m, z, m, \ldots, z, m) & \text{if } k \text{ is odd.} \end{cases}$$

Since m is distinct from all n_i, m can never appear in any occurrence of z or $\bar{z}$ in the $A\text{-code}$ of w^ℓ. So $\theta_z^A(w^\ell) = \ell$, whereas $\theta_{\bar{z}}^A(w^\ell) = 0$ since z is generic. Consequently,

$$f_z^A(w^\ell) = \theta_z^A(w^\ell) - \theta_{\bar{z}}^A(w^\ell) = \ell,$$

which shows that f_z^A is unbounded.

Second, consider the case $A \cong B$ and fix a choice of isomorphism. Let z be generic. It holds that $|A| = |B| \geq 3$ since $A * B$ is not the infinite dihedral group. Consider the swap isomorphism s interchanging the factors A and B, where we use the fixed isomorphism from before to identify A and B with each other. Then the application of s to any element g interchanges the $A\text{-code}$ and $B\text{-code}$ of g with each other. This implies that the sum $\theta_z^A + \theta_z^B$ is invariant under s and consequently the sum $f_z^A + f_z^B$ is invariant under s as well. Again, by Lemma 4.8 f_z^A and f_z^B define quasimorphisms invariant under all factor automorphisms and so does their sum $f_z^A + f_z^B$. According to Lemma 3.4 this means that $f_z^A + f_z^B$ is invariant under a full set of representatives for $\mathrm{Out}(A * B)$. Again, by Lemma 2.19 we see that the homogenisation $\bar{f}_z^A + \bar{f}_z^B$ is invariant under all automorphisms of $A * B$. It remains to verify unboundedness.

For this let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ be non-trivial such that $a_1 \neq a_2$ and $b_1 \neq b_2$. Pick a non-zero number $m \in \mathbb{N}$ distinct from all $n_i \in \mathbb{N}$, where $z = (n_1, \ldots, n_k)$. As before, we set

$$w_0 = (a_1 b_1)^{n_1} (a_2 b_2)^{n_2} (a_1 b_1)^{n_3} (a_2 b_2)^{n_4} \ldots (a_s b_s)^{n_k},$$

where s is 1 or 2 depending on whether k is odd or even. We set

$$w = \begin{cases} w_0 & \text{if } k \text{ is even,} \\ w_0(a_2b_2)^m & \text{if } k \text{ is odd.} \end{cases}$$

Then the A-code and B-code of w agree and are given by

$$A\text{-code}(w) = B\text{-code}(w) = \begin{cases} (n_1, \dots, n_k) = z & \text{if } k \text{ is even,} \\ (n_1, \dots, n_k, m) = (z, m) & \text{if } k \text{ is odd.} \end{cases}$$

Since m is distinct from all n_i, m can never appear in any occurrence of z or $\bar{z}$ in the A-code and B-code of w^ℓ. As in the first case, $\theta_z^A(w^\ell) = \theta_z^B(w^\ell) = \ell$, whereas $\theta_{\bar{z}}^A(w^\ell) = \theta_{\bar{z}}^B(w^\ell) = 0$ since z is generic. Consequently,

$$f_z^A(w^\ell) + f_z^B(w^\ell) = \theta_z^A(w^\ell) + \theta_z^B(w^\ell) - \theta_{\bar{z}}^A(w^\ell) - \theta_{\bar{z}}^B(w^\ell) = 2\ell,$$

which shows that $f_z^A + f_z^B$ is unbounded and therefore its homogenisation is the desired unbounded Aut-invariant quasimorphism on $A * B$.

Finally, let us verify that the space of homogeneous Aut-invariant quasimorphisms on $A * B$ that vanish on letters is infinite-dimensional. Let $r \in \mathbb{N}$ and let $z_1, \dots, z_r$ be generic tuples. Choose z_{r+1} be a 3-tuple whose entries are distinct non-zero natural numbers and do not appear in any of the z_i; then z_{r+1} is generic. It follows from the above construction of the word w for z_{r+1} in both cases that any linear combination of the associated code quasimorphisms $f_{z_1}^A + f_{z_1}^B, \dots, f_{z_r}^A + f_{z_r}^B$ vanishes on all powers of w. It follows that the same holds for any linear combination of their homogenisations $\bar{f}_{z_1}^A + \bar{f}_{z_1}^B, \dots, \bar{f}_{z_r}^A + \bar{f}_{z_r}^B$. Thus, $\bar{f}_{z_{r+1}}^A + \bar{f}_{z_{r1}}^B$ is not contained in the subspace spanned by the first r quasimorphisms. Clearly, the homogenisation of any code quasi-morphism vanishes on all letters of $A * B$. Since $r \in \mathbb{N}$ was arbitrary, it follows that the space of homogeneous Aut-invariant quasimorphisms on $A * B$ that vanish on letters cannot have finite dimension. □

Example 4.12. Let $G = A * B$ for $A = \mathbb{Z}/5$ and $B \neq 1$ be freely indecomposable. Consider

$$g = a^4ba^2ba^2ba^3bababa^3bababa^2ba^2ba^2b$$

for non-trivial $a \in A, b \in B$. Then the A-tuple of g is $(a^4, a^2, a^2, a^3, a, a, a^3, a, a, a^2, a^2, a^2)$. It follows that $A\text{-code}(g) = (1, 2, 1, 2, 1, 2, 3)$. Note that $A\text{-code}(g^{-1}) = (3, 2, 1, 2, 1, 2, 1)$. For $z = (1, 2, 3)$ we have $\theta_z^A(g) = 1$ and $\theta_z^A(g^{-1}) = 0$. So $f_z^A(g) = 1$. In fact, for all $n \in \mathbb{N}$ we have $\theta_z^A(g^n) = n$, $\theta_z^A(g^{-n}) = 0$ and so $f_z^A(g^n) = n$. This implies that the homogenisation $\bar{f}_z^A$ that is Aut-invariant for $B \notin \{\mathbb{Z}/2, \mathbb{Z}\}$ by Proposition 4.11 satisfies $\bar{f}_z^A(g) > 0$ and is unbounded on powers of g.

5 Weighted code quasimorphisms

If one of the factors of a free product $A * B$ of freely indecomposable groups happens to be infinite cyclic, the code quasimorphisms above are in general not Aut-invariant since they are not necessarily invariant with respect to transvections. Thus, we need to modify our original construction to deal with infinite cyclic factors. Afterwards we will follow steps similar to the previous section in order to establish their Aut-invariance.

Lemma 5.1. *Let B be a non-trivial group and let w be any word in $\mathbb{Z} * B$ such that w only contains letters of the same sign from $\mathbb{Z}$ and starts and ends with a non-zero letter from $\mathbb{Z}$. Then its unique reduced form w' starts and ends with a letter from $\mathbb{Z}$ with that given sign. Moreover, the sum over all letters in w belonging to the factor $\mathbb{Z}$ remains the same in its reduced form w'.*

Proof. Any word in the free product is brought to its reduced form by successively eliminating trivial letters and replacing two adjacent letters from the same factor by their product in that factor. The sum of all letters from $\mathbb{Z}$ stays the same because any two adjacent letters of $\mathbb{Z}$ are always replaced by their sum throughout the reduction process. The only way to encounter an elimination of the first letter $a_1 \in \mathbb{Z}$ or the last letter $a_n \in \mathbb{Z}$ during the reduction process would be by the occurrence of $-a_1$ or $-a_n$. This is not possible since a_1 and a_n are non-zero and all letters have the same sign by assumption. □

Definition 5.2 (Weighted $\mathbb{Z}$-code)**.** Let B be freely indecomposable. Write $g \in \mathbb{Z}*B$ in reduced form. Let $(a_1, \ldots, a_k)$ be the $\mathbb{Z}$-tuple of g. We define a tuple $(x_1, \ldots, x_\ell)$ of non-zero natural numbers as follows. Consider the successive subsequences of maximal length in $(a_1, \ldots, a_k)$ consisting of integers all of the same sign. For the i-th such sequence, we define x_i to be the absolute value of the sum of integers in that sequence. We call the tuple $(x_1, \ldots, x_\ell)$ the *weighted $\mathbb{Z}$-code* of g.

Example 5.3. Let B be a non-trivial group and let $b_i \in B$ be non-trivial elements. Then the reduced word

$$w = 7b_1(-2)b_2(-4)b_3(-1)b_4 9b_5 2b_6(-3)$$

has the $\mathbb{Z}$-tuple $(7, -2. -4, -1, 9, 2, -3)$ which yields the weighted $\mathbb{Z}$-code $(7, 7, 11, 3)$.

Definition 5.4 (Weighted code quasimorphisms)**.** Let $z = (n_1, \ldots, n_k)$ be a tuple of non-zero natural numbers. We set $\theta_z^{\mathbb{Z}} \colon \mathbb{Z} * B \to \mathbb{Z}_{\geq 0}$ to count the number of *disjoint* occurrences of z as a tuple of consecutive numbers inside the weighted $\mathbb{Z}$-code of $g \in \mathbb{Z} * B$. Define the *weighted code quasimorphism*

$$f_z^{\mathbb{Z}} \colon \mathbb{Z} * B \to \mathbb{Z} \quad \text{by} \quad f_z^{\mathbb{Z}}(g) = \theta_z^{\mathbb{Z}}(g) - \theta_z^{\mathbb{Z}}(g^{-1})$$

for all $g \in \mathbb{Z} * B$. Note that we again have $\theta_z^{\mathbb{Z}}(g^{-1}) = \theta_{\bar{z}}^{\mathbb{Z}}(g)$ for all g.

Lemma 5.5. *Let z be a non-empty tuple of non-zero natural numbers. Then the counting function $\theta_z^{\mathbb{Z}} \colon \mathbb{Z} * B \to \mathbb{Z}_{\geq 0}$ satisfies*

1. $\theta_z^{\mathbb{Z}}(g^{-1}) = \theta_{\bar{z}}^{\mathbb{Z}}(g)$ *for all $g \in \mathbb{Z} * B$,*
2. $|\theta_z^{\mathbb{Z}}(w_1 w_2) - \theta_z^{\mathbb{Z}}(w_1) - \theta_z^{\mathbb{Z}}(w_2)| \leq 2$ *for all reduced words w_1, w_2 in $\mathbb{Z} * B$ for which $w_1 w_2$ is a reduced word.*

Moreover, $f_z^{\mathbb{Z}} \colon \mathbb{Z} * B \to \mathbb{Z}$ defined for all $g \in \mathbb{Z} * B$ by $f_z^{\mathbb{Z}}(g) = \theta_z^{\mathbb{Z}}(g) - \theta_z^{\mathbb{Z}}(g^{-1})$ is a quasimorphism of defect $D(f_z^{\mathbb{Z}}) \leq 30$.

Proof. First, recall that the reduced form of g^{-1} is obtained by inverting the reduced form of g, which amounts to reversing the order and inverting all letters. This means to obtain the weighted $\mathbb{Z}$-code of g^{-1} one needs to reverse the one of g. Consequently, counting the number of disjoint occurrences of z in the weighted $\mathbb{Z}$-code of g^{-1} amounts to counting the disjoint occurrences of the reversed tuple $\bar{z}$ in the weighted $\mathbb{Z}$-code of g itself. This proves the first part.

Second, let w_1, w_2 be written as reduced words with $\mathbb{Z}$-tuples given by $(n_1, \dots, n_k)$ for w_1 and $(m_1, \dots, m_\ell)$ for w_2 for integers n_i, m_j. Let $(x_1, \dots, x_{k'})$ and $(y_1, \dots, y_{\ell'})$ be the weighted $\mathbb{Z}\text{-codes}$ of w_1 and w_2. By assumption there is no cancellation or reduction in the product of their reduced expressions representing w_1w_2. That means that the last letter of w_1 and the first letter of w_2 belong to different factors. Then

$$\text{weighted } \mathbb{Z}\text{-code}(w_1w_2) = \begin{cases} (x_1, \dots, x_{k'}, y_1, \dots, y_{\ell'}) & \text{if } \operatorname{sgn}(n_k) \neq \operatorname{sgn}(m_1), \\ (x_1, \dots, x_{k'-1}, x_{k'} + y_1, y_2, \dots, y_{\ell'}) & \text{if } \operatorname{sgn}(n_k) = \operatorname{sgn}(m_1). \end{cases}$$

If $\operatorname{sgn}(n_k) \neq \operatorname{sgn}(m_1)$, then $\theta_z^{\mathbb{Z}}(w_1w_2) \in \{\theta_z^{\mathbb{Z}}(w_1) + \theta_z^{\mathbb{Z}}(w_2), \theta_z^{\mathbb{Z}}(w_1) + \theta_z^{\mathbb{Z}}(w_2) + 1\}$ since at most one of the disjoint occurrences of z can involve numbers that do not lie completely in the weighted $\mathbb{Z}\text{-code}$ of either w_1 or w_2.

If $\operatorname{sgn}(n_k) = \operatorname{sgn}(m_1)$, then $\theta_z^{\mathbb{Z}}(w_1w_2) \geq \theta_z^{\mathbb{Z}}(w_1) + \theta_z^{\mathbb{Z}}(w_2) - 2$ since only one occurrence of z in the weighted $\mathbb{Z}\text{-code}$ of w_1 and w_2 can involve $x_{k'}$ or y_1 respectively. Moreover, if an occurrence of z in the weighted $\mathbb{Z}\text{-code}$ of w_1w_2 involves $x_{k'} + y_1$, then all other occurrences of z are either fully contained in the weighted $\mathbb{Z}\text{-code}$ of w_1 or fully contained in the weighted $\mathbb{Z}\text{-code}$ of w_2. Thus, $\theta_z^{\mathbb{Z}}(w_1w_2) \leq \theta_z^{\mathbb{Z}}(w_1) + \theta_z^{\mathbb{Z}}(w_2) + 1$.

In both cases we conclude that

$$|\theta_z^{\mathbb{Z}}(w_1w_2) - \theta_z^{\mathbb{Z}}(w_1) - \theta_z^{C}(w_2)| \leq 2.$$

It follows from Lemma 2.17 that $f_z^{\mathbb{Z}}$ is a quasimorphism of defect at most 30. □

Lemma 5.6. *For all non-empty tuples z the weighted code quasimorphism $f_z^{\mathbb{Z}} \colon \mathbb{Z} * B \to \mathbb{Z}$ is invariant under factor automorphisms and transvections of the first factor.*

Proof. It is immediate from the definition that the weighted $\mathbb{Z}\text{-code}$ of any element in the free product is invariant under factor automorphisms. Let x be a generator of the infinite cyclic factor in $\mathbb{Z} * B$. Any transvection is defined to be the identity on letters from B and maps $x \to xy$ or $x \to yx$ for some non-trivial element $y \in B$. Let us consider the transvection φ uniquely specified by $x \to xy$ and show that the weighted $\mathbb{Z}\text{-code}$ of any element in $\mathbb{Z} * B$ is invariant under φ. Then it immediately follows that $\theta_z^{\mathbb{Z}}$ and $f_z^{\mathbb{Z}}$ are invariant under φ. The argument for transvections of the second kind will follow analogously to the one we present now.

Let $w \in \mathbb{Z} * B$ be a reduced word such that its weighted $\mathbb{Z}\text{-code}$ has length one. This means that all letters from $\mathbb{Z}$ in the reduced expression of w have the same sign and the weighted $\mathbb{Z}\text{-code}$ is given by the image of w under the factor projection $\mathbb{Z} * B \to \mathbb{Z}$. Note that this factor projection is invariant with respect to φ and so the weighted $\mathbb{Z}\text{-code}$ of $\varphi(w)$ agrees with the one of w. There cannot be any cancellations of letters from $\mathbb{Z}$ occurring.

Let us do a preliminary calculation to visualise the general case more easily. Let k, ℓ be non-zero natural numbers and $b \in B$ non-trivial. Then

$$\begin{aligned} \varphi(x^k b x^{-\ell}) &= \varphi(x)^k b \varphi(x)^{-\ell} = (xy)^k b (y^{-1}x^{-1})^\ell = xy \dots xyxyby^{-1}x^{-1}y^{-1}x^{-1} \dots y^{-1}x^{-1}, \\ \varphi(x^{-k} b x^{\ell}) &= \varphi(x)^{-k} b \varphi(x)^{\ell} = (y^{-1}x^{-1})^k b (xy)^\ell = y^{-1}x^{-1} \dots y^{-1}x^{-1}bxy \dots xy. \end{aligned}$$

This shows that the letter from B separating the positive and negative powers of x either remains b or is a conjugate of b in B after applying φ.

Let $w \in \mathbb{Z} * B$ be a reduced word with weighted $\mathbb{Z}\text{-code}$ of length $k \geq 2$. In w we formally gather all consecutive occurrences of powers of x of the same sign and call these sub-words w_i for $i = \{1, \dots, k\}$. That is, we write the reduced word w uniquely as a product of reduced words as

$$w = w_1 b_1 w_2 b_2 \dots w_{k-1} b_{k-1} w_k,$$

where the $b_i \in B$ are non-trivial and the w_i are of of maximal length such that all letters from $\mathbb{Z}$ inside any w_i have the same sign. Moreover, in this decomposition w_1 ends with a letter from $\mathbb{Z}$, w_n starts with a letter from $\mathbb{Z}$ and all other w_i start and end with letters from $\mathbb{Z}$. By the maximality of w_i all letters from $\mathbb{Z}$ occurring in w_i have different signs from the ones occurring in w_{i+1} for all i.

We apply φ to w and obtain an a priori not necessarily reduced word, which we rewrite in the previous block form as

$$\varphi(w) = \varphi(w_1)b_1\varphi(w_2)b_2 \dots \varphi(w_{k-1})b_{k-1}\varphi(w_k) = w_1'b_1'w_2'b_2' \dots w_{k-1}'b_{k-1}'w_k',$$

where $b_i' = yby^{-1}$ if the letters from $\mathbb{Z}$ change sign from positive to negative at b_i and $b_i' = b_i$ if they change from negative to positive. Moreover, all letters from $\mathbb{Z}$ inside any w_i' have the same sign again, w_1' ends with a letter from $\mathbb{Z}$, w_n' starts with a letter from $\mathbb{Z}$ and all other w_i' start and end with letters from $\mathbb{Z}$.

We observe that when bringing $\varphi(w)$ to its reduced form there cannot be any cancellations of the letters b_i'. This is because by Lemma 5.1 the letters that are adjacent to b_i will always remain letters from $\mathbb{Z}$ after the reduction procedure of all w_i'. Indeed, replacing all w_i' by their reduced forms w_i'' we see that the product

$$w'' = w_1''b_1'w_2''b_2 \dots w_{k-1}''b_{k-1}'w_k''$$

is the reduced representative of $\varphi(w)$ since the letters adjacent to the b_i' are always letters from $\mathbb{Z}$. Consequently, no cancellations in between letters of different signs from $\mathbb{Z}$ can occur when bringing $\varphi(w)$ to its reduced form. The reduced words w_i'' have the same weighted $\mathbb{Z}$-code as the original w_i for all i. Therefore, the weighted $\mathbb{Z}$-code of $\varphi(w)$ agrees with the weighted $\mathbb{Z}$-code of w. □

Proposition 5.7. *Let B be a freely indecomposable group which is not infinite cyclic. Then for any generic tuple of natural numbers z the homogenisation $\bar{f}_z^{\mathbb{Z}} \colon \mathbb{Z} * B \to \mathbb{R}$ of the quasimorphism $f_z^{\mathbb{Z}}$ is an unbounded Aut-invariant quasimorphism on $\mathbb{Z} * B$. Moreover, the space of homogeneous Aut-invariant quasimorphisms on $\mathbb{Z} * B$ that vanish on letters has infinite dimension.*

Proof. By Lemma 5.5 $f_z^{\mathbb{Z}}$ is a quasimorphism that is invariant under factor automorphisms and transvections according to Lemma 5.6. Images of these automorphisms generate the outer automorphism group $\mathrm{Out}(\mathbb{Z} * B)$ by Lemma 3.4 . Thus, $f_z^{\mathbb{Z}}$ is invariant under a full set of representatives of all outer automorphisms and so by Lemma 2.19 the homogenisation $\bar{f}_z^{\mathbb{Z}}$ is invariant under $\mathrm{Aut}(\mathbb{Z} * B)$. It remains to check that it is unbounded, which is equivalent to $f_z^{\mathbb{Z}}$ itself being unbounded.

Since z is generic, $z = (n_1, \dots, n_k)$ for some $k \geq 3$ where all $n_i \in \mathbb{N}$ are non-zero. Let $b \in B$ be non-trivial and m a strictly positive integer number distinct from all n_i. Set $w \in \mathbb{Z} * B$ to be

$$w = \begin{cases} n_1b(-n_2)bn_3b(-n_4)\dots b(-n_k)b & \text{if } k \text{ is even,} \\ n_1b(-n_2)bn_3b(-n_4)\dots b(-n_k)bmb & \text{if } k \text{ is odd.} \end{cases}$$

The weighted $\mathbb{Z}$-code of w is given by

$$\text{weighted } \mathbb{Z}\text{-code}(w) = \begin{cases} (n_1, \dots, n_k) = z & \text{if } k \text{ is even,} \\ (n_1, \dots, n_k, m) = (z, m) & \text{if } k \text{ is odd.} \end{cases}$$

Since w starts and ends with letters belonging to different factors, the reduced expression of w^ℓ is the ℓ-fold product of the word w for all $\ell \in \mathbb{N}$. Moreover, since the first and last letter from $\mathbb{Z}$ in w have different signs the weighted $\mathbb{Z}$-code of w^ℓ is

$$\text{weighted } \mathbb{Z}\text{-code}(w^\ell) = \begin{cases} (z, z, \ldots, z) & \text{if } k \text{ is even,} \\ (z, m, z, m, \ldots, z, m) & \text{if } k \text{ is odd.} \end{cases}$$

Since m is distinct from all n_i, m cannot appear in any occurrence of z or $\bar{z}$ inside the weighted $\mathbb{Z}$-code of w^ℓ. So $\theta_z^{\mathbb{Z}}(w^\ell) = \ell$, whereas $\theta_{\bar{z}}^{A}(w^\ell) = 0$ since z is generic. Consequently,

$$f_z^{\mathbb{Z}}(w^\ell) = \theta_z^{\mathbb{Z}}(w^\ell) - \theta_{\bar{z}}^{\mathbb{Z}}(w^\ell) = \ell,$$

which shows that $f_z^{\mathbb{Z}}$ is unbounded.

Finally, let us verify that the space of homogeneous Aut-invariant quasimorphisms on $\mathbb{Z} * B$ that vanish on letters is infinite-dimensional. Clearly, the homogenisation of any weighted code quasimorphism vanishes on all letters of $\mathbb{Z} * B$. Let $r \in \mathbb{N}$ and let $z_1, \ldots, z_r$ be generic tuples. Choose z_{r+1} be a 3-tuple whose entries are distinct non-zero natural numbers and do not appear in any of the z_i; then z_{r+1} is generic. It follows from the above construction of the word w for z_{r+1} that any linear combination of $f_{z_1}^{\mathbb{Z}}, \ldots, f_{z_r}^{\mathbb{Z}}$ vanishes on all powers of this w. It follows that the same holds for any linear combination of their homogenisations $\bar{f}_{z_1}^{\mathbb{Z}}, \ldots, \bar{f}_{z_r}^{\mathbb{Z}}$. Thus, $\bar{f}_{z_{r+1}}^{\mathbb{Z}}$ is not contained in the subspace spanned by the first r quasimorphisms. Since, $r \in \mathbb{N}$ was arbitrary, it follows that the space of homogeneous Aut-invariant quasimorphisms on $\mathbb{Z} * B$ that vanish on letters cannot have finite dimension. □

Remark 5.8. Proposition 5.7 does not hold for $B = \mathbb{Z}$. Indeed, $\bar{f}_z^{\mathbb{Z}}$ does no longer need to be invariant under $\mathrm{Aut}(\mathbb{Z} * B)$ since the weighted $\mathbb{Z}$-code is in general not invariant under transvections of the factor B.

For example, consider $z = (4, 3, 2, 1)$ and denote the standard generators of $\mathbb{Z} * B$ by $x \in \mathbb{Z}$ and $y \in B$. Following the proof of Proposition 5.7 the element $w = x^4 y x^{-3} y x^2 y x^{-1} y$ with weighted $\mathbb{Z}$-code of $(4, 3, 2, 1)$ satisfies $f_z^{\mathbb{Z}}(w^\ell) = \ell$ for all $\ell \in \mathbb{N}$. Let $\varphi \in \mathrm{Aut}(\mathbb{Z} * B)$ be the transvection of the second factor defined by $\varphi(x) = x$ and $\varphi(y) = x^3 y$. Then we have $\varphi(w) = x^7 y^2 x^5 y x^2 y$. So the weighted $\mathbb{Z}$-code of $\varphi(w)$ is the tuple with a single entry equal to 14. Thus, $f_z^{\mathbb{Z}}(\varphi(w^\ell)) = f_z^{\mathbb{Z}}(\varphi(w)^\ell) = 0$ for all $\ell \in \mathbb{N}$. So $\bar{f}_z^{\mathbb{Z}}$ evaluates non-trivially on w, but trivially on $\varphi(w)$, which means that $\bar{f}_z^{\mathbb{Z}}$ is not invariant under $\mathrm{Aut}(\mathbb{Z} * B)$.

Moreover, this example can be used to show that the sum of the two weighted code quasimorphisms $\bar{f}_z^{\mathbb{Z}} + \bar{f}_z^{B}$ is not necessarily invariant under $\mathrm{Aut}(\mathbb{Z} * B)$ either, which contrasts the situation of free factors that are not infinite cyclic in Proposition 4.11 (2).

5.1 Naturality of (weighted) code quasimorphisms with respect to inclusions

Let us now show that (weighted) code quasimorphisms satisfy a naturality condition with respect to inclusions. Using this property will only become necessary once, in the last part of the proof of Theorem 9.27.

Lemma 5.9. *Let A, B, C, D be freely indecomposable groups such that $C \leq A$, $D \leq B$ are subgroups. Let $G = A * B$ and let $H = C * D$ inside G. Let z be any tuple of positive integers and consider the counting functions $\theta_z^A \colon G \to \mathbb{Z}$ and $\theta_z^C \colon H \to \mathbb{Z}$. Then the restriction to the subgroup $H \leq G$ satisfies $(\theta_z^A)|_H = \theta_z^C$. Consequently, we have $(f_z^A)|_H = f_z^C$ for the corresponding code quasimorphisms which implies $(\bar{f}_z^A)|_H = \bar{f}_z^C$ for their homogenisations.*

Proof. Any word in H is reduced if and only if its image in G is reduced. Therefore, for any $h \in H$ with C-tuple given by $(c_1, \dots, c_k)$ where $c_i \in C$ for all i we have that the A-tuple of h is given by $(c_1, \dots, c_k)$ as well. Consequently, we have $A\text{-code}(h) = C\text{-code}(h)$ for all $h \in H$. So $(\theta_z^A)|_H = \theta_z^C$ for all tuples z of positive integers and the rest of the statement follows as well. □

6 Applications of code quasimorphisms

Theorem 6.1. *Let $G = A * B$ be the free product of two non-trivial freely indecomposable groups A and B. Assume G is not the infinite dihedral group. Then G admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms, all of which vanish on single letters.*

Proof. The space of homogeneous Aut-invariant quasimorphisms on $\mathbb{Z} * \mathbb{Z}$ has infinite dimension by [BrMa19, Theorem 2]. Inverting both generators of the factors defines an automorphism which inverts all letters in $\mathbb{Z} * \mathbb{Z}$. So any homogeneous Aut-invariant quasimorphism on $\mathbb{Z} * \mathbb{Z}$ vanishes on all letters. For all other free products of two factors Proposition 4.11 and Proposition 5.7 imply the existence of infinitely many linearly independent homogeneous Aut-invariant quasimorphisms, all of which vanish on letters. □

Corollary 6.2. *Let $G = A * B$ be the free product of two non-trivial freely indecomposable groups and assume G is not the infinite dihedral group. Then there exists a stably unbounded Aut-invariant norm on G.*

Proof. Let $A*B$ be a free product of two freely indecomposable groups which is not the infinite dihedral group. By Theorem 6.1 there exist unbounded Aut-invariant quasimorphisms on $A*B$ that are bounded on all letters. Since $A * B$ is generated by letters, the result follows from Lemma 2.9. □

Remark 6.3. If neither A nor B is infinite cyclic, then Corollary 6.2 can also be deduced from the result given in [Mar20, Lemma 4.4] together with the explicit description of the automorphism group given in Section 3.

Corollary 6.4. *Let $H \to G \to A*B$ be an extension of a free product of freely indecomposable groups A and B by a group H. Assume that H is a characteristic subgroup of G and $A * B$ is not the infinite dihedral group. Then the space of homogeneous Aut-invariant quasimorphisms on G is infinite-dimensional.*

Proof. The space of homogeneous Aut-invariant quasimorphisms on $A * B$ has infinite dimension by Theorem 6.1. Therefore, the result follows from Lemma 2.23. □

Corollary 6.5. *Let $G_1 *_H G_2$ be a free product of groups G_1, G_2 amalgamated over a common subgroup H which is proper and central in both G_1 and G_2. If G_1/H and G_2/H are freely indecomposable and not both equal to $\mathbb{Z}/2$, the space of homogeneous Aut-invariant quasimorphisms on $G_1 *_H G_2$ is infinite-dimensional.*

Proof. By assumption $H \neq G_1$ and $H \neq G_2$ and so H equals the center of $G_1 *_H G_2$. As such it is a characteristic subgroup of $G_1 *_H G_2$. Furthermore, $\frac{G_1 *_H G_2}{H} \cong \frac{G_1}{H} * \frac{G_2}{H}$. By assumption $\frac{G_1}{H} * \frac{G_2}{H}$ is not isomorphic to the infinite dihedral group and the result follows from Corollary 6.4 above. □

Example 6.6. For $q \geq 3$, the Hecke groups $H_q \cong \mathbb{Z}/2 * \mathbb{Z}/q$ admit infinitely many linearly independent homogeneous Aut-invariant quasimorphisms by Theorem 6.1.

Example 6.7. By Corollary 6.5 the space of homogeneous Aut-invariant quasimorphisms on $\mathrm{SL}(2, \mathbb{Z})$ is infinite-dimensional, since $\mathrm{SL}(2, \mathbb{Z})$ is the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$.

Example 6.8. The braid group B_3 admits infinitely many linearly independent homogeneous Aut-invariant quasimorphism as well by Corollary 6.4. Indeed, B_3 is the central extension of $\mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$ by $\mathbb{Z}$. For more details see Appendix B.

Example 6.9. Let $G_{p,q} = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ be the free product of two copies of the integers amalgamated over inclusions $\iota_1, \iota_2 \colon \mathbb{Z} \to \mathbb{Z}$ which are multiplication by p and q. For coprime choices of p and q these are the so-called knot groups $K_{p,q}$ arising as the fundamental group of the complement of torus knots. Then $G_{p,q}$ admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms if $\min\{|p|, |q|\} \geq 2$ and $\max\{|p|, |q|\} \geq 3$. We have seen in Example 2.14 that this is no longer true for $p = q = 2$.

7 General case of free products

Recall, that for a free product G the group $\mathrm{Aut}^0(G)$ is the subgroup of its automorphism group generated by factor automorphisms, transvections and partial conjugations.

Lemma 7.1. *Let $k \geq 3$ and let $G = G_1 * G_2 * \cdots * G_k$ be a free product of freely indecomposable factors such that G_i is not infinite cyclic for $i \geq 3$. Then the kernel $\ker(p)$ of the projection $p \colon G \to G_1 * G_2$ is invariant under $\mathrm{Aut}^0(G)$.*

Proof. The kernel $\ker p$ is normally generated by letters belonging to $G_3, \ldots, G_k$. Thus, $\ker(p)$ is clearly invariant under all factor automorphisms. Since G_i is not infinite cyclic for $i \geq 3$, all transvections act trivially on letters of G_i. Moreover, since $\ker(p)$ is a normal subgroup, it is invariant under all partial conjugations. These three classes of automorphisms generate $\mathrm{Aut}^0(G)$ and so $\ker(p)$ is invariant under $\mathrm{Aut}^0(G)$. □

Lemma 7.2. *Let $k \geq 3$ and let $G = G_1 * G_2 * \cdots * G_k$ be a free product of freely indecomposable factors such that G_i is not infinite cyclic for $i \geq 3$. Let $p \colon G \to G_1 * G_2$ be the projection and let $\psi \colon G_1 * G_2 \to \mathbb{R}$ be an Aut-invariant quasimorphism. Then $\psi \circ p \colon G \to \mathbb{R}$ is an Aut^0-invariant quasimorphism on G.*

Proof. Clearly, $\psi \circ p$ is a quasimorphism. Its Aut^0-invariance follows from Lemma 7.1 together with the Aut-invariance of ψ. □

Theorem 7.3. *Let $G = G_1 * \cdots * G_k$ be a free product of freely indecomposable groups G_i where $k \geq 2$. Assume that at most two factors are infinite cyclic and there exists $j \in \{1, \ldots, k\}$ such that $G_j \not\cong \mathbb{Z}/2$. Then G admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms, all of which vanish on single letters.*

Proof. After reordering the free factors we may assume that G_i is not infinite cyclic for $i \geq 3$ and that $G_1 \not\cong \mathbb{Z}/2$. Let ψ be an Aut-invariant quasimorphism on $G_1 * G_2$ that is homogeneous, unbounded and vanishes on single letters. The existence of such a quasimorphism follows from Theorem 6.1 . By Lemma 7.2 the composition $\psi \circ p$, where $p \colon G \to G_1 * G_2$ denotes the projection map, is Aut^0-invariant on G. By Lemma 3.3 the index of $\mathrm{Aut}^0(G)$ in $\mathrm{Aut}(G)$ is finite with a system of coset representatives given by automorphisms permuting the factors. We denote this system by $\{\sigma_i\}_{i\in I}$. Therefore, by Lemma 2.21 the quasimorphism $\widehat{\psi \circ p}$ defined for $g \in G$ by $\widehat{\psi \circ p}(g) = \sum_i \psi(p(\sigma_i(g)))$ is invariant under the whole automorphism group of G. It is homogeneous as well. Let us verify that it is unbounded.

Let $g \in G_1 * G_2$. For any $i \in I$ and $j \in \{1, \ldots, k\}$ we have on the level of letters that $\sigma_i(G_j) = G_{\sigma_i(j)}$. So, if $\{\sigma_i(1), \sigma_i(2)\} \neq \{1, 2\}$ then $p(\sigma_i(g))$ is either trivial or just a single letter of G_1 or G_2. However, we know that ψ vanishes on single letters and so $\psi(p(\sigma_i(g)))$ vanishes for all σ_i that fail to satisfy $\{\sigma_i(1), \sigma_i(2)\} = \{1, 2\}$. So let $J \leq I$ be the subset consisting of σ_j satisfying $\{\sigma_j(1), \sigma_j(2)\} = \{1, 2\}$. Any $\sigma_j \in J$ descends to an automorphism $\bar{\sigma_j}$ of $G_1 * G_2$ under p. However, we know that ψ is invariant under all automorphisms of $G_1 * G_2$, so in particular under the ones of the form $\bar{\sigma_j}$. Consequently, we calculate for $g \in G_1 * G_2$ that

$$\widehat{\psi \circ p}(g) = \sum_{i\in I} \psi(p(\sigma_i(g))) = \sum_{j\in J} \psi(p(\sigma_j(g))) = \sum_{j\in J} \psi(\bar{\sigma_j}(g))) = |J|\psi(g),$$

where $|J|$ denotes the cardinality of the set J. Note that $|J| \geq 1$ since it contains the representative of the class representing the identity.

This means that for any Aut-invariant quasimorphism ψ on $G_1 * G_2$ that vanishes on letters the Aut-invariant quasimorphism $\widehat{\psi \circ p}$ on G restricts to a linear multiple of ψ on the subgroup $G_1 * G_2 \leq G$. Additionally, $\widehat{\psi \circ p}$ vanishes on all letters as well. Consequently, the space of homogeneous Aut-invariant quasimorphisms on G is infinite dimensional since the space of homogeneous Aut-invariant quasimorphisms on $G_1 * G_2$ is infinite dimensional by Theorem 6.1. Moreover, all of these homogeneous quasimorphisms can be chosen to vanish on letters. □

Example 7.4. $B_3 *_{\mathbb{Z}} B_3$, the free product of B_3 with itself amalgamated over their common center generated by the Garside element, admits infinitely many linearly independent unbounded Aut-invariant quasimorphisms. To prove this we cannot apply Corollary 6.5 directly since $B_3/\mathbb{Z}$ is not freely indecomposable. The center of $B_3 *_{\mathbb{Z}} B_3$ is again generated by the Garside element of each of the factors. This fits into the short exact sequence

$$\mathbb{Z} \to B_3 *_{\mathbb{Z}} B_3 \to (B_3/\mathbb{Z}) * (B_3/\mathbb{Z}).$$

Finally, $(B_3/\mathbb{Z}) * (B_3/\mathbb{Z}) = \mathrm{PSL}(2, \mathbb{Z}) * \mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3 * \mathbb{Z}/2 * \mathbb{Z}/3$. So Theorem 7.3 applies. Then the statement for $B_3 *_{\mathbb{Z}} B_3$ follows from Lemma 2.23.

8 Graph products of abelian groups

8.1 Definitions

Definition 8.1 (Graph)**.** A finite graph Γ is a pair (V, E) consisting of a non-empty finite set of vertices V and a finite set of unoriented edges E where an unoriented edge is a subset of V of cardinality two. So two distinct vertices have at most one edge between them and no vertex has an edge to itself. A graph Γ is called *complete* if there exists an edge between all pairs of distinct vertices in Γ. For a subset $X \subset V$ we denote by Γ_X the subgraph of Γ spanned by X.

Definition 8.2. Let $\Gamma = (V, E)$ be a graph.

1. The link of $v \in V$ is defined to be $lk(v) = \{x \in V : (v, x) \in E\}$.

2. The star of $v \in V$ is defined to be $st(v) = lk(v) \cup \{v\}$.

Definition 8.3. A relation $\sim$ on a set X is called *reflexive* if $x \sim x$ for all $x \in X$. Further, $\sim$ is called *transitive* if $x \sim y$, $y \sim z$ implies that $x \sim z$ for all $x, y, z \in X$.

Definition 8.4 ([CoGu09, Def. 5.4])**.** A preorder is a relation that is reflexive and transitive. We define two preorders on any graph $\Gamma = (V, E)$ by:

$$\begin{aligned} v \leq w &\quad \text{if and only if} \quad lk(v) \subset st(w), \\ v \leq_s w &\quad \text{if and only if} \quad st(v) \subset st(w). \end{aligned}$$

The preorder $\leq$ has already been defined before in [ChVo09] with the proof of transitivity given in [ChVo09, Lemma 2.1].

Definition 8.5 (Graph product)**.** Let $\Gamma = (V, E)$ be a finite graph. Let $\{G_v\}_{v \in V}$ be a set of non-trivial groups. The graph product defined by Γ and $\{G_v\}_{v \in V}$ is the group

$$W(\Gamma, \{G_v\}_{v \in V}) = (*_{v \in V} G_v)/N,$$

where N is the normal subgroup generated by all $[G_x, G_y]$ for $x, y \in V$ such that $\{x, y\} \in E$.

Definition 8.6. Let $W(\Gamma, \{G_v\}_{v \in V})$ be a graph product of groups. If all vertex groups are $\mathbb{Z}/2$, then $W(\Gamma, \{G_v\}_{v \in V})$ is the *right angled Coxeter group* on the graph Γ. If all vertex groups are infinite cyclic, then $W(\Gamma, \{G_v\}_{v \in V})$ is the *right angled Artin group* on Γ which will be denoted by R_Γ from now onwards.

Definition 8.7 (Truncated subgroup)**.** Let $\Gamma = (V, E)$ be a graph and $\{G_v\}_{v \in V}$ be a collection of finitely generated abelian groups. Let $V' \subset V$ be a non-empty subset of vertices and let Γ' be the subgraph of Γ spanned by V'. Then $W(\Gamma', \{G_v\}_{v \in V'})$ is called a *truncated subgroup* of $W(\Gamma, \{G_v\}_{v \in V})$. If $W(\Gamma, \{G_v\}_{v \in V})$ decomposes as the cartesian product of the truncated subgroup $W(\Gamma', \{G_v\}_{v \in V'})$ spanned by $V' \subset V$ and the one spanned by $V - V'$, then $W(\Gamma', \{G_v\}_{v \in V'})$ is called a *direct truncated subgroup* of $W(\Gamma, \{G_v\}_{v \in V})$.

A group that is isomorphic to $\mathbb{Z}/p^k$ where k is a positive integer and p is a prime is called *primary*. Recall that finitely generated abelian groups can be uniquely decomposed into a cartesian product with primary and infinite cyclic factors. Thus, one sees that for any graph product of finitely generated abelian groups $W(\Gamma, \{G_v\}_{v \in V})$ there is a graph $\Gamma' = (V', E')$ together with a collection of groups $\{G'_v\}_{v \in V'}$, where each G'_v is primary or infinite cyclic, such that $W(\Gamma, \{G_v\}_{v \in V}) \cong W(\Gamma', \{G'_v\}_{v \in V'})$. Namely, $\Gamma' = (V', E')$ is obtained by replacing each $v \in V$ by the complete graph with vertices corresponding to the generators of the primary and infinite cyclic factors of G_v. We say that Γ' is the graph *expanding* Γ in this case. Clearly, for a graph product of finite abelian groups $W(\Gamma, \{G_v\}_{v \in V})$ there is a collection $\{G'_v\}_{v \in V'}$ consisting only of primary groups such that $W(\Gamma, \{G_v\}_{v \in V}) \cong W(\Gamma, \{G'_v\}_{v \in V'})$.

We simplify notation by denoting $W(\Gamma, \{G_v\}_{v \in V})$ as W_Γ for the case where the set $\{G_v\}_{v \in V}$ only consists of primary or infinite cyclic groups. In W_Γ we will abuse notation to identify each vertex of Γ with a chosen generator of the cyclic group G_v. Therefore, vertices can be considered as elements in W_Γ.

The next lemma shows that the above replacement of $W(\Gamma, \{G_v\}_{v \in V})$ by $W(\Gamma, \{G'_v\}_{v \in V'})$ does not create new direct truncated subgroups that are free products of primary or infinite cyclic groups.

Lemma 8.8. *Let $\Gamma = (V, E)$ be a graph and let $W(\Gamma, \{G_v\}_{v \in V})$ be a graph product of finitely generated abelian groups. Let $\Gamma' = (V', E')$ and $\{G'_v\}_{v \in V'}$ where each G'_v is primary or infinite* *cyclic be the graph expanding $W(\Gamma, \{G_v\}_{v\in V})$ such that $W(\Gamma', \{G'_v\}_{v\in V'} \cong W(\Gamma, \{G_v\}_{v\in V})$. Assume that for some $k \geq 2$ there is a direct truncated subgroup in $W(\Gamma', \{G'_v\}_{v\in V'})$ of the form $H = G_{v_1} * \cdots * G_{v_k}$, then H is a direct truncated subgroup of $W(\Gamma, \{G_v\}_{v\in V})$ generated by the same vertex set.*

Proof. For $v_1 \in V'$ there is a corresponding vertex $w_1 \in V$ such that v_1 originates from w_1 when replacing the finitely generated abelian vertex group G_{w_1} by its primary and infinite cyclic factors. Assume there was another generator w inside the vertex group G_{w_1} yielding another vertex w in the graph Γ'. Then since v_1 and w originate from the same vertex group they commute in $W(\Gamma', \{G'_v\}_{v\in V'}$ as well and so $w \notin \{v_1, \dots, v_k\}$. Then w belongs to the truncated subgroup spanned by $V' - \{v_1, \dots, v_k\}$. Since H is a direct truncated subgroup, this means that w commutes with all v_i. However, since v_1 and w originate from the same vertex group, they also satisfy $st(w) = st(v_1)$. This is a contradiction since $v_2 \in st(w)$ but $v_2 \notin st(v_1)$. Therefore, the vertex group G_{w_1} is the same as G_{v_1} to begin with. Similarly, this holds for all v_i for $i \geq 2$ and $\{w_1, \dots, w_k\}$ span H as a direct truncated subgroup in $W(\Gamma, \{G_v\}_{v\in V})$. □

The question of when a graph product decomposes as a direct product of truncated subgroups has been solved in [Gre90] for graph products with general vertex groups. We only give a version restricted to our case of finitely generated abelian vertex groups here, where we assume that the vertex set of the graph in consideration has cardinality at least two since truncated subgroups have non-trivial underlying vertex set by definition.

Theorem 8.9 ([Gre90, Thm. 3.34]). *Let $\Gamma = (V, E)$ be a graph with $|V| \geq 2$ and let $\{G_v\}_{v\in V}$ be a collection of finitely generated abelian groups. Then $W(\Gamma, \{G_v\}_{v\in V})$ has non-trivial center if and only if $W(\Gamma, \{G_v\}_{v\in V})$ is a direct product of truncated subgroups at least one of which has non-trivial center.*

Inductively applying this result immediately yields the following description of graph products of finitely generated abelian groups.

Corollary 8.10. *Let Γ be a graph and $\{G_v\}_{v\in V}$ be a collection of finitely generated abelian groups. Then $W(\Gamma, \{G_v\}_{v\in V})$ has non-trivial center if and only if there exists a vertex $v \in V$ such that $st(v) = V$.*

Proof. Clearly, any $v \in V$ such that $st(v) = V$ belongs to the center of $W(\Gamma, \{G_v\}_{v\in V})$. Conversely, if the center of $W(\Gamma, \{G_v\}_{v\in V})$ is non-trivial, then it decomposes into a direct product of truncated subgroups at least one of which has a non-trivial center. Since truncated subgroups are graph products themselves , we can iterated this procedure until we end up with a subgroup over a single vertex $v \in V$ that is a direct factor of $W(\Gamma, \{G_v\}_{v\in V})$. This implies $st(v) = V$. □

8.2 Automorphisms of graph products

Recall that for a graph product W_Γ where all vertex groups are primary or infinite cyclic we perceive the vertices of Γ as elements in W_Γ by identifying them with a generator of their respective vertex group. There are the following four families of automorphisms for graph products W_Γ for which each vertex group is primary or infinite cyclic (see [CoGu09, p.1–2] combined with [CoGu09, Prop. 5.5]).

1. Every isomorphism of graphs $\gamma\colon \Gamma \to \Gamma$ such that $G_v = G_{\gamma(v)}$ for all $v \in V$ induces an automorphism of W_Γ. Such automorphisms are called *labelled graph automorphisms*.

2. Let $v \in V$ and $m \in \mathbb{Z}$ such that $\gcd(m, |G_v|) = 1$. If $|G_v| = \infty$, then $m = \pm 1$. Then we set the *factor automorphism* $\phi_{v,m}$ to be the unique automorphism of W_Γ defined for $z \in V$ by

$$\phi_{v,m}(z) = \begin{cases} v^m & z = v, \\ z & z \neq v. \end{cases}$$

3. Let $v, w \in V$ be two distinct vertices. A *dominated transvection* is an automorphism $\tau_{v,w}$ defined having one of the two forms:

   (a) $|G_v| = \infty$, $v \leq w$ and

$$\tau_{v,w}(z) = \begin{cases} vw & z = v, \\ z & z \neq v, \end{cases}$$

   (b) $|G_v| = p^k$, $|G_w| = p^\ell$, $v \leq_s w$ and

$$\tau_{v,w}(z) = \begin{cases} vw^q & z = v, \\ z & z \neq v, \end{cases}$$

   where $q = \max\{1, p^{\ell-k}\}$ and p is prime.

4. Let $v \in V$ and let K be the vertex set of a connected component of $\Gamma - st(v)$. Define the *partial conjugation* by v on K by

$$\sigma_{K,v}(z) = \begin{cases} vzv^{-1} & z \in K, \\ z & z \notin K. \end{cases}$$

Theorem 8.11 ([CoGu09, p.2]). *Let $G = W_\Gamma$ be a graph product where each vertex group is primary or infinite cyclic. Then $\mathrm{Aut}(G)$ is generated by the above four families of automorphisms.*

Definition 8.12. Let $G = W_\Gamma$. Then the subgroup of $\mathrm{Aut}(G)$ generated by automorphisms of type 2–4 above is denoted by $\mathrm{Aut}^0(G)$.

Note that this definition agrees with Definition 3.2 for free products where all factors are primary or infinite cyclic.

Lemma 8.13. *[Mar20, Prop 2.1] $\mathrm{Aut}^0(W_\Gamma)$ is a finite index subgroup of $\mathrm{Aut}(W_\Gamma)$. A set of coset representatives is given by labelled graph automorphisms.*

9 Aut-invariant quasimorphisms on graph products

From now onwards we will only consider graph products of finitely generated abelian groups. Recall, that by Lemma 2.9 the existence of any unbounded Aut-invariant quasimorphism on a finitely generated group implies the existence of an unbounded Aut-invariant norm on that group. Thus, as corollaries of our constructions we will reprove the existence of unbounded Aut-invariant norms shown in [Mar20].

9.1 Lower cones

Definition 9.1. Let $\Gamma = (V, E)$ be a graph and let $X \subset V$ be a subset of vertices. The map $R_X : W_\Gamma \to W_{\Gamma_X}$ defined by

$$R_X(z) = \begin{cases} z & z \in X, \\ e & z \notin X. \end{cases}$$

is called a *standard retraction* onto the full subgraph of Γ generated by X. We denote the kernel of R_X by K_X.

Lemma 9.2 ([Mar20, Lemma 3.3])**.** *Let $X \subset V$. The group K_X is invariant under factor automorphisms and partial conjugations.*

Definition 9.3. Let $\Gamma = (V, E)$ be a graph and let $W(\Gamma, \{G_v\}_{v \in V})$ be a graph product of a family of groups $\{G_v\}_{v \in V}$ each of which is primary or infinite cyclic. Let $\leq_\tau$ be the relation on V defined for $v, w \in V$ by $v \leq_\tau w$ if and only if the dominated transvection $\tau_{v,w}$ is well-defined.

Definition 9.3 above is the central definition of this subsection. For the remainder of this subsection we always implicitly assume that $\Gamma = (V, E)$ is a graph and $W_\Gamma = W(\Gamma, \{G_v\}_{v \in V})$ is a graph product of a family of groups $\{G_v\}_{v \in V}$ each of which is primary or infinite cyclic. Recall that a partial order on a set X is a preorder $\precsim$ for which $x = y$ whenever $x \precsim y$ and $y \precsim x$ for all $x, y \in X$.

Lemma 9.4 ([Mar20, Lemma 3.4])**.** *The relation $\leq_\tau$ is a preorder on V.*

Definition 9.5. Define a relation $\sim_\tau$ on V by setting $v \sim_\tau w$ if and only if $v \leq_\tau w$ and $w \leq_\tau v$ for $v, w \in V$. Since $\leq_\tau$ is a preorder, $\leq_\tau$ defines a partial order on the equivalence classes of $\sim_\tau$ in V.

Lemma 9.6 ([Mar20, p.9])**.** *Let $\Gamma = (V, E)$ be a graph $M \subset V$ be an equivalence class of $\sim_\tau$. Then W_{Γ_M} is either finite and abelian, or free abelian, or a free group.*

Definition 9.7 (Lower Cone)**.** Let Y be a set and $\leq$ a relation on Y. A subset $X \subset Y$ is called a *lower cone* if for all $t \in X$, $s \in Y$ the relation $s \leq t$ implies that $s \in X$.

Lemma 9.8. *Unions and intersections of lower cones are lower cones.*

Proof. Let $X_i \subset Y$ be lower cones with respect to $\leq$ for $i \in I$. Let $t \in \bigcup_{i \in I} X_i$ and $s \in Y$ be given such that $s \leq t$. Then there exists $j \in I$ such that $t \in X_j$. Since X_j is a lower cone, $s \in X_j$ and consequently $s \in \bigcup_{i \in I} X_i$. Therefore, $\bigcup_{i \in I} X_i$ is a lower cone.

Let $t \in \bigcap_{i \in I} X_i$ and $s \in Y$ be such that $s \leq t$. Then $t \in X_i$ for all $i \in I$. Since every X_i is a lower cone, we have that $s \in X_i$ for all $i \in I$ and therefore $s \in \bigcap_{i \in I} X_i$. Therefore, $\bigcap_{i \in I} X_i$ is a lower cone. □

Example 9.9. The complement of any minimally chosen subset of vertices $V' \subset V$ whose removal disconnects the graph $\Gamma = (V, E)$ is a lower cone with respect to $\leq_\tau$. In fact, let $C_1, \ldots, C_k$ be the vertex sets of the connected components of $\Gamma_{V - V'}$. Then for all i each vertex $v \in C_i$ satisfies $st(v) \subset C_i \cup V'$. But since V' is chosen minimally with respect to the property that its removal disconnects Γ, the link of each vertex $w \in V'$ contains vertices from at least two distinct connected components C_i and C_j. Thus, no vertex $v \in V'$ satisfies $v \leq_\tau w$ for any $w \in V - V' = C_1 \cup \cdots \cup C_k$.

Lemma 9.10. *Let $v \in V$ such that $st(v) = V$ and assume that G_v is finite for all v. Then the equivalence class $[v]$ of $\sim_\tau$ is maximal with respect to $\leq_\tau$.*

Proof. Let $w \in V$ be such that $v \leq_\tau w$. Then $V = st(v) = st(w)$. Since $\tau_{v,w}$ is well-defined, $|G_v| = p^k$ and $|G_w| = p^\ell$ for a prime p. So $\tau_{w,v}$ is well defined as well. Thus $w \leq_\tau v$ and therefore $w \sim_\tau v$. □

Lemma 9.11. *Let R_Γ be a right angled Artin group. Let $v \in V$ such that $st(v) = V$. Then the equivalence class $[v]$ of $\sim_\tau$ is maximal with respect to $\leq_\tau$.*

Proof. Let $w \sim_\tau v$ for some $w \neq v$. Since $st(v) = V$, we have that $w \in lk(v)$ and so $v \in lk(w)$. Moreover, since $v \leq_\tau w$, we have $V - \{v\} = lk(v) \subset st(w)$. Thus, $st(w) = V$ for all $w \sim_\tau v$. Trivially, $lk(z) \subset V = st(v)$ for all $z \in V$ and so any $z \in V$ satisfies $z \leq_\tau v$. □

Lemma 9.12. *Let $\Gamma = (V, E)$ be a graph in which no vertex $v \in V$ satisfies $st(v) = V$. If W_Γ decomposes as a product of direct truncated subgroups $W_\Gamma \cong W_{\Gamma_{V_1}} \times W_{\Gamma_{V_2}}$, then both V_1 and V_2 are lower cones with respect to $\leq_\tau$.*

Proof. By the definition of direct truncated subgroups V is the disjoint union $V = V_1 \cup V_2$. Because of the direct product decomposition $W_\Gamma \cong W_{\Gamma_{V_1}} \times W_{\Gamma_{V_2}}$ each generator $v \in V_1$ commutes with each generator $w \in V_2$ and vice versa. If there were $v \in V_1$, $w \in V_2$ such that $lk(v) \subset st(w)$, then $V_2 \subset st(w)$ and so $st(w) = V$ which is a contradiction. So there do not exist any vertices $v \in V_1$, $w \in V_2$ such that $v \leq_\tau w$ which implies that V_2 is a lower cone. By symmetry, V_1 is a lower cone as well. □

Definition 9.13. Let $\Gamma = (V, E)$ be a graph and $M \subset V$ be a subset. Define

$$L_M = \{v \in V \mid [v, w] \neq e \text{ for all } w \in M\}.$$

By definition $M \cap L_M = \emptyset$ for all M. Equivalently, $L_M = \{v \in V \mid st(v) \subset V - M\}$ and so $L_M = V - (\bigcup_{w \in M} st(w))$.

The next lemma is crucial for finding a lower cone in a graph such that the associated graph product of the lower cone decomposes as a non-trivial free product.

Lemma 9.14. *The set $M \cup L_M$ is a lower cone with respect to $\leq_\tau$ for every minimal equivalence class $M \subset V$ of $\sim_\tau$. Moreover, if G_v is finite for all $v \in V$, then L_M is itself a lower cone for any subset $M \subset V$.*

Proof. Let $w \in V$. If $w \leq_\tau v$ for $v \in M$, then $w \in M$ by the minimality of M with respect to $\leq_\tau$. Now assume that $w \leq_\tau v$ for $v \in L_M$. If w has finite order, then $w \leq_s v$. Therefore, $st(w) \subset st(v) \subset V - M$ and so $w \in L_M$. Finally, if w has infinite order, then we have $lk(w) \subset st(v) \subset V - M$. So either $w \in M$ or $st(w) \subset V - M$. Thus, $w \in M \cup L_M$. □

Lemma 9.15. *Let M, N be two equivalence classes such that W_{Γ_N} is not a free group of rank $k \geq 2$. If $N \cap L_M \neq \emptyset$, then $N \subset L_M$.*

Proof. Let $a \in M$ and $x \in N \cap L_M$. Then $x \in L_M$ implies that $a \notin lk(x)$. Since $M \cap L_M = \emptyset$, it follows that $a \notin st(x)$. Any $y \sim_\tau x$ satisfies $lk(y) \subset st(x)$. So $a \notin lk(y)$. Since $W_{\Gamma_N} \not\cong F_k$, the vertices x and y are connected by an edge. Therefore, $y \notin M$. So $st(y) \in V - M$. Since $a \in M$ was chosen arbitrarily and $y \sim_\tau x$ was chosen arbitrarily, it follows that $N \subset L_M$. □

Lemma 9.16 ([Mar20, Lemma 3.5])**.** *If $X \subset V$ is a lower cone with respect to $\leq_\tau$, then K_X is invariant under* $\mathrm{Aut}^0(W_\Gamma)$.

The following two lemma follows immediately from Lemma 9.16.

Lemma 9.17. *Let $X \subset V$ be a lower cone with respect to $\leq_\tau$ that is invariant under labelled graph automorphisms. Then K_X is a characteristic subgroup of W_Γ.* □

Lemma 9.18. *Let $\psi \colon W_\Gamma \to W_\Gamma$ be a labelled graph automorphism. Then ψ preserves the relation $\leq_\tau$. Consequently, ψ preserves equivalence classes of $\sim_\tau$.*

Proof. Clearly, ψ satisfies $|G_v| = |G_{\psi(v)}|$ for all $v \in V$. Moreover, $\psi(st(v)) \subset st(\psi(v))$ for all $v \in V$. By applying ψ^{-1} it follows that $\psi(st(v)) = st(\psi(v))$ for all $v \in V$. Therefore, ψ preserves $\leq_\tau$ and so ψ preserves equivalence classes of $\sim_\tau$. □

Corollary 9.19. *Let M, N be equivalence classes of $\sim_\tau$ and let $\psi \colon W_\Gamma \to W_\Gamma$ be a labelled graph automorphism. Then $\psi(M) \cap N \neq \emptyset$ if and only if $\psi(M) = N$.*

Lemma 9.20. *Let $X \subset V$ be a lower cone with respect to $\leq_\tau$ and Y be the orbit of X under the action of labelled graph automorphisms. Then K_Y is a characteristic subgroup of W_Γ.*

Proof. By Lemma 9.18 the image of any lower cone under a labelled graph automorphism is a lower cone. By Lemma 9.8 the union of lower cones is a lower cone itself and so the result follows from Lemma 9.17. □

Proposition 9.21. *Let $X \subset V$ be a lower cone with respect to $\leq_\tau$ that is additionally invariant under all labelled graph automorphisms. If W_{Γ_X} is a free product of $k \geq 2$ freely indecomposable groups $G_1, \ldots, G_k$ such that at least one of the G_i is not $\mathbb{Z}/2$ and at most two are infinite cyclic, then W_Γ admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms.*

Proof. By Lemma 9.17 the kernel of the retraction map $R_X \colon W_\Gamma \to W_{\Gamma_X}$ is characteristic. Consequently, by Lemma 2.23 any Aut-invariant quasimorphism of W_{Γ_X} gives rise to an Aut-invariant quasimorphism on W_Γ. The result follows from Theorem 7.3. □

Example 9.22. Let Γ and Λ be the graphs pictured below. Consider the right angled Artin groups R_Γ and R_Λ. Recall that in this case for any two vertices a, b the relation $a \leq_\tau b$, which is by definition equivalent to the existence of the dominated transvection $\tau_{a,b}$, is satisfied if and only if $lk(a) \subset st(b)$.

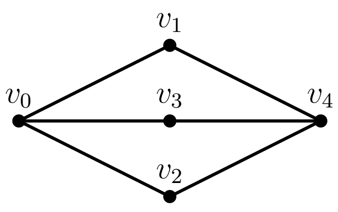


Figure 1: graph Γ

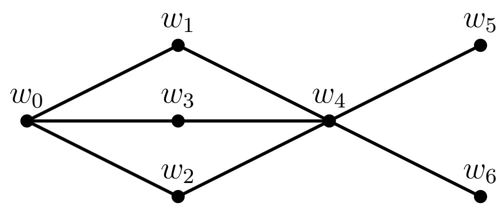


Figure 2: graph Λ

In R_Γ the sets of vertices $X = \{v_0, v_4\}$ and $Y = \{v_1, v_2, v_3\}$ both form minimal equivalence classes of $\sim_\tau$. Since the cardinality of X and Y differs, both are preserved by any labelled graph automorphism by Lemma 9.19 . Since R_{Γ_X} is the free product of two infinite cyclic groups, Proposition 9.21 applies.

In R_Λ the vertices w_0 and w_4 are not equivalent with respect to $\sim_\tau$. In fact, R_Λ contains four equivalence classes $A = \{w_0\}$, $B = \{w_1, w_2, w_3\}$, $C = \{w_4\}$ and $D = \{w_5, w_6\}$, where C is the only equivalence class that is not minimal. Let $X = A \cup D$ and $Y = A \cup C \cup D$. Then X and Y are both lower cones that are invariant under labelled graph automorphisms since such automorphism preserve the cardinality of the star of each vertex. However, since R_{Λ_X} is the group of rank three, Proposition 9.21 does not apply to X. Instead, Proposition 9.21 applies to Y because $R_{C \cup D} \cong \mathbb{Z} \times F_2$ is freely indecomposable since it has non-trivial center.

9.2 Aut-invariant quasimorphisms on some classes of right angled Artin groups

Proposition 9.23. *Let Γ be a finite graph and R_Γ be the right angled Artin group on Γ. Assume that one of the following two conditions is satisfied*

- *There is a minimal equivalence class M of $\sim_\tau$ such that $R_{\Gamma_M} \cong F_2$,*
- *No equivalence class N of $\sim_\tau$ satisfies $R_{\Gamma_N} \cong F_k$ for $k \geq 2$.*

Then, either R_Γ is a free abelian group or the space of homogeneous Aut-invariant quasimorphisms on R_Γ has infinite dimension.

Proof. First, assume that M is a minimal equivalence class of $\sim_\tau$ such that $R_{\Gamma_M} \cong F_2$. Then M is a lower cone of R_Γ by Lemma 9.16. This implies that any unbounded Aut-invariant quasimorphism on F_2 gives rise to an unbounded Aut^0-invariant quasimorphism on R_Γ.

Let σ be a labelled graph automorphism of Γ and let ψ be an unbounded Aut-invariant quasimorphism on R_{Γ_M}. According to Lemma 9.19 either $\sigma(M) = M$, in which case σ descends to an automorphism of R_{Γ_M}, or $\sigma(M) \cap M = \emptyset$. Since the labelled graph automorphisms $\{\sigma_i\}$ form a set of coset representative for $\mathrm{Aut}^0(R_\Gamma)$ in $\mathrm{Aut}(R_\Gamma)$ according to Lemma 8.13 , the quasimorphism $\widehat{\psi \circ p}$ defined for $g \in R_\Gamma$ by $\widehat{\psi \circ p}(g) = \sum_i \psi(p(\sigma_i(g)))$ is invariant under the whole automorphism group $\mathrm{Aut}(R_\Gamma)$ by Lemma 2.21.

Thus, for all $g \in R_{\Gamma_M} \leq R_\Gamma$ it holds that

$$\widehat{\psi \circ p}(g) = \sum_i \psi(p(\sigma_i(g))) = |J|\psi(g),$$

where J is the subset of labelled graph automorphisms satisfying $\sigma_i(M) = M$. Clearly, $|J| \geq 1$. So the Aut-invariant quasmorphism $\widehat{\psi \circ p}$ restricted to the subgroup $R_{\Gamma_M} \leq R_\Gamma$ is just a linear multiple of ψ. Then the result follows from Theorem 6.1 , which in this case is due to [BrMa19, Thm. 2].

Second, assume that no equivalence class N of $\sim_\tau$ satisfies $R_{\Gamma_N} \cong F_k$ for $k \geq 2$. Then every equivalence class N satisfies $R_{\Gamma_N} \cong \mathbb{Z}^k$ for some $k \geq 1$ by Lemma 9.6. Let M be a minimal equivalence class. By Lemma 9.14 $M \cup L_M$ is a lower cone. If $L_M = \emptyset$, then M commutes with all other elements. By Lemma 9.10 M is maximal and so $V - M$ is a lower cone. Proceed with choosing a minimal equivalence class in $V - M$ and iterate the construction. This process will either yield a nontrivial N such that L_N is non-trivial or implies that R_Γ is a free abelian group itself. So we may now assume that L_M is non-trivial and we consider the lower cone $M \cup L_M$.

According to Lemma 9.15 any equivalence class N of $\sim_\tau$ with $N \cap L_M \neq \emptyset$ is fully contained in L_M. Let N be a minimal equivalence class in L_M. Then $M \cup N$ is a lower cone. Consequently, $p\colon R_\Gamma \to R_{\Gamma_M} * R_{\Gamma_N}$ is an Aut^0-equivariant projection. Moreover, $R_{\Gamma_M} * R_{\Gamma_N} \cong \mathbb{Z}^\ell * \mathbb{Z}^k$ for some $k, \ell \geq 1$. Let $\psi\colon R_{\Gamma_M} * R_{\Gamma_N} \to \mathbb{Z}$ be an Aut-invariant quasimorphism. Let σ be labelled graph automorphism of Γ. According to Lemma 9.19 either $\sigma(M) = M$, in which case

σ descends to an automorphism of R_{Γ_M}, or $\sigma(M) \cap M = \emptyset$. Since the labelled graph automorphisms $\{\sigma_i\}$ form a set of coset representative for $\mathrm{Aut}^0(R_\Gamma)$ in $\mathrm{Aut}(R_\Gamma)$ according to Lemma 8.13 , the quasimorphism $\widehat{\psi \circ p}$ defined for $g \in R_\Gamma$ by $\widehat{\psi \circ p}(g) = \sum_i \psi(p(\sigma_i(g)))$ is invariant under the whole automorphism group $\mathrm{Aut}(R_\Gamma)$ by Lemma 2.21.

Thus for any Aut-invariant quasimorphism ψ on $R_{\Gamma_M} * R_{\Gamma_N}$ and for all $g \in R_{\Gamma_M} * R_{\Gamma_N} \leq R_\Gamma$ we calculate

$$\widehat{\psi \circ p}(g) = \sum_i \psi(p(\sigma_i(g))) = |J|\psi(g),$$

where J is the subset of labelled graph automorphisms satisfying $\{\sigma(M), \sigma(N)\} = \{M, N\}$. Clearly, $|J| \geq 1$. So restricted to the subgroup $R_{\Gamma_M} * R_{\Gamma_N} \leq R_\Gamma$ the Aut-invariant quasmorphism is just a linear multiple of ψ. Then the statement follows from Theorem 6.1. $\square$

Theorem 9.24. *Let $\Gamma = (V, E)$ be a finite graph with $|V| \geq 2$ and such that no two distinct vertices $x, y \in V$ satisfy $lk(v) \subset st(v)$. Then the space of homogeneous Aut-invariant quasimorphisms on the right angled Artin group R_Γ has infinite dimension.*

Proof. Since any equivalence class M of R_Γ consisting of at least two vertices admits nontrivial transvections by definition, every equivalence class of $\sim_\tau$ consists of a single vertex. Since there are two vertices that do not commute, R_Γ cannot be free abelian. Thus, the statement follows from Proposition 9.23. $\square$

9.3 Freely indecomposable graph products

Lemma 9.25. *Let $A * B$ be a free product of two non-trivial groups. If $x \in A * B$ has finite order then x is a conjugate of a letter belonging to A or B.*

Proof. Let $x \in A * B$ be an element that has finite order and is not conjugate to a letter and such that the reduced form $w = c_1 \dots c_k$ of x has minimal length k among all elements of finite order in $A * B$ that are not conjugate to a letter. If c_1 and c_k were letters from different factors, the order of x would be infinite. So c_1 and c_k belong to the same factor. Let a be the letter representing the product of c_k with c_1 in that factor. Then the reduced form of $c_1^{-1} x c_1$ is given by $c_2 \dots c_{k-1} a$, which has length $k - 1$ contradicting the minimality of x. Consequently, the result follows. $\square$

Lemma 9.26. *Let Γ be a connected graph of primary groups. Then W_Γ is freely indecomposable.*

Proof. Assume $W_\Gamma \cong G_1 * G_2$. By Lemma 9.25 every element of finite order is conjugate to a letter of G_1 or G_2. Let $v \in \Gamma$ be a vertex generating a primary group G_v. Without loss of generality we can assume that v is conjugate to a non-trivial letter of G_1. That is, v can be written as a reduced word $a g_1 a^{-1}$, where $g_1 \in G_1$ and a is a reduced word.

Let w be adjacent to v. Then w is conjugate to a non-trivial letter by Lemma 9.25. We will prove by contradiction that w is a conjugate of a letter of G_1 as well. Assume that w was conjugate to a letter of G_2. Then w can be written as a reduced word as $b g_2 b^{-1}$ where $g_2 \in G_2$. Let c be the reduced form of $a^{-1} b$. Since v and w are adjacent, they commute. Then

$$\begin{aligned} 1 = a^{-1}[v, w]a = a^{-1}[a g_1 a^{-1}, b g_2 b^{-1}]a &= g_1 a^{-1} b g_2 b^{-1} a g_1^{-1} a^{-1} b g_2^{-1} b^{-1} a \\ &= g_1 c g_2 c^{-1} g_1^{-1} c g_2 c^{-1}. \end{aligned}$$

If c is the empty word, this yields a contradiction since g_1 and g_2 are non-trivial letters belonging to different factors of the free product and so their commutator is non-trivial. Therefore, c is non-trivial. If the last letter of c belongs to G_2, then we define c_0 to be the reduced word obtained by omitting this last letter of c. Otherwise we set $c_0 = c$. Then $cg_2c^{-1} = c_0g_2c_0^{-1}$ where the latter expression is reduced. Consequently, the expression $c_0g_2^{-1}c_0^{-1}$ is reduced and represents $cg_2^{-1}c^{-1}$. We compute

$$1 = a^{-1}[v, w]a = g_1cg_2c^{-1}g_1^{-1}cg_2c^{-1} = g_1c_0g_2c_0^{-1}g_1^{-1}c_0g_2^{-1}c_0^{-1}.$$

Again, this would yield a contradiction if c_0 represented the identity. However, if c_0 was to begin with a letter from G_2, the expression $g_1c_0g_2c_0^{-1}g_1^{-1}c_0g_2^{-1}c_0^{-1}$ would be reduced and therefore be non-trivial as well. Therefore, c_0 begins with a non-trivial letter $x \in G_1$ and its reduced form has length ≥ 2. Let c_1 be the non-trivial reduced word obtained by omitting the first letter of c_0. Let y be the letter from G_1 representing the product of g_1 and x or empty if $g_1x = 1$. Then the product of reduced words $yc_1g_2c_1^{-1}g_1^{-1}c_1g_2^{-1}c_1^{-1}x^{-1}$ is a non-trivial reduced word itself and represents the same element as $g_1c_0g_2c_0^{-1}g_1^{-1}c_0g_2^{-1}c_0^{-1}$. This is a contradiction.

Consequently, w is a conjugate of a letter belonging to G_1. By induction it follows that all vertex groups belong to the conjugacy class of G_1 in W since Γ is connected. Since the vertex groups generate W_Γ, it follows that W_Γ is completely contained in the conjugacy class of G_1. Then it lies in the kernel of the projection onto the factor G_2 and so G_2 is the trivial group. It follows that W_Γ is freely indecomposable. □

9.4 Aut-invariant quasimorphisms on graph products of finite abelian groups

Recall that a direct truncated subgroup of a graph product W on a graph $\Gamma = (V, E)$ is spanned by a subset of the vertex set $V' \subset V$ such that W decomposes as a cartesian product of the graph products on the subsets V' and $V - V'$. Moreover, recall that by definition a graph product of finite groups W is finite if and only if the underlying graph is complete since W has a non-trivial free product as a subgroup otherwise. Finally, recall that we proved the existence of Aut-invariant quasimorphisms on all free products of finite groups where not all factors are equal to $\mathbb{Z}/2$ in Theorem 7.3. In the proof of the following proposition we use the notion of lower cones to produce Aut-invariant quasimorphisms on graph products of finite groups from Aut-invariant quasimorphisms on free products.

Theorem 9.27. *Let $\Gamma = (V, E)$ be a finite graph that is not complete. Let $\{G_v\}_{v\in V}$ a family of finite abelian groups and let $W(\Gamma, \{G_v\}_{v\in V})$ be their graph product. Let Z_k be the k-fold free product of groups of order two $Z_k = \mathbb{Z}/2 * \cdots * \mathbb{Z}/2$. Assume that $W(\Gamma, \{G_v\}_{v\in V})$ does not decompose as a product $G_1 \times \cdots \times G_\ell$ for $\ell \geq 1$ where each G_i is a direct truncated subgroup that is isomorphic to some Z_k for $k \geq 2$ or finite abelian. Then the space of homogeneous Aut-invariant quasimorphisms on $W(\Gamma, \{G_v\}_{v\in V})$ has infinite dimension.*

Proof. We can replace Γ and $\{G_v\}_{v\in V}$ with a graph Γ' and a collection of primary abelian groups $\{G'_v\}_{v\in V}$ such that $W(\Gamma, \{G_v\}_{v\in V}) \cong W(\Gamma', \{G'_v\}_{v\in V}) =: W_{\Gamma'}$. If Γ' has more than one connected component, then $W_{\Gamma'}$ is the free product of the graph products associated to the connected components of Γ'. By Lemma 9.26 each of those connected components is itself freely indecomposable and by assumption $W(\Gamma, \{G_v\}_{v\in V})$ is not a free product of groups of order two. So Theorem 7.3 applies and the result follows.

Therefore, we may assume that Γ' is a connected graph. Moreover, recall that by Lemma 8.8 if Z_k is a direct truncated subgroup of $W(\Gamma', \{G'_v\}_{v\in V})$, then Z_k is a direct truncated subgroup in $W(\Gamma, \{G_v\}_{v\in V})$. We will outline an iterative procedure proving the following claim.

Claim. If $W(\Gamma, \{G_v\}_{v\in V})$ is not finite, there exists a lower cone L in Γ' and $n \geq 2$ such that $W_{\Gamma'_L}$ is a free product of $W_1, \dots, W_n$, the groups generated by the connected components of L, such that $W_1, \dots, W_n$ are not infinite cyclic and are not all equal to $\mathbb{Z}/2$.

If $st(v) = V$ for some $v \in V$, then the equivalence class $[v] \subset V$ is maximal with respect to $\leq_\tau$ by Lemma 9.10. Thus, $V - [v]$ is a lower cone with respect to $\leq_\tau$ and $W_{\Gamma'} \cong W_{\Gamma'_{[v]}} \times W_{\Gamma'-[v]}$ where $\Gamma' - [v]$ is the subgraph of Γ' spanned by all vertices $v \in V - [v]$. According to Lemma 9.6 the group $W_{\Gamma'_{[v]}}$ is finite and abelian. We then proceed by considering the graph $\Gamma' - [v]$. Therefore, we will assume in the following that no $v \in V$ satisfies $st(v) = V$.

From now onwards let M be an equivalence class which is minimal with respect to $\leq_\tau$. Again, Lemma 9.6 implies that M generates a finite abelian group $W_{\Gamma'_M}$ in $W_{\Gamma'}$, since all G'_v are finite. Moreover, $st(v) = st(w)$ for all $v, w \in M$. We consider the set of vertices $M \cup L_M$ which form a lower cone with respect to $\leq_\tau$ according to Lemma 9.14. The set L_M is non-empty since $st(v) \neq V$ for $v \in M$. Since no vertex $v \in M$ shares an edge with a vertex $w \in L_M$, the group $W_{M\cup L_M}$ decomposes as a free product $W_M * W_{L_M}$.

Since no free factor can be the infinite cyclic group this proves the claim unless $W_M \cong Z/2$ and all connected components of L_M consist of single vertices with vertex groups equal to $\mathbb{Z}/2$. Then $M = \{x\}$ and $L_M = \{y_1, \dots, y_\ell\}$ for some $\ell \geq 1$ and both are lower cones with respect to $\leq_\tau$ according to Lemma 9.14. In fact, since all equivalence classes generate finite abelian groups, this implies that each vertex y_i is its own equivalence class and therefore minimal since L_M is a lower cone. All vertices in $V - (M \cup L_M)$ commute with x by definition of L_M. If there was $z \in V - (M \cup L_M)$ such that $[z, y_i] \neq 1$ for some i, then $x, z \in L_{\{y_i\}}$. In this case $\{y_i\} \cup L_{\{y_i\}}$ would be a lower cone for which $W_{\{y_i\}\cup L_{\{y_i\}}}$ decomposes as a free product of two freely indecomposable factors which are not all equal to $\mathbb{Z}/2$ since one connected component contains an edge.

Otherwise, we are in the situation where x and all y_i commute with all other vertices. Then $M \cup L_M$ generates a direct truncated subgroup in Γ' that is isomorphic to Z_k. By Lemma 8.8 Z_k comes from a direct truncated subgroup spanned by the same vertex set in $W(\Gamma, \{G_v\}_{v\in V})$. By Lemma 9.12 the complement $V - (M \cup L_M)$ is a lower cone and we restart the iterative procedure using the graph $\Gamma_{V-(M\cup L_M)}$. If W_Γ is not finite, this process either terminates with a lower cone as specified in the claim or yields a decomposition of W_Γ into direct truncated subgroups all of which are of the form Z_k for some $k \geq 2$. However, the latter is impossible by assumption. This proves the claim.

The claim implies by Lemma 9.16 and Lemma 7.1 there exists an Aut^0-equivariant map $p\colon W_{\Gamma'} \to W_a * W_b$ for some $a, b \in \{1, \dots, n\}$ such that $W_a * W_b \ncong D_\infty$. Let A and B be minimal equivalence classes in W_a and W_b respectively. We distinguish two cases.

First, consider the case where Γ_A and Γ_B are not both equal to $\mathbb{Z}/2$. These two equivalence classes are lower cones themselves, which means that the projection $q\colon W_{\Gamma'} \to W_{\Gamma_A} * W_{\Gamma_B}$ is Aut^0-equivariant. Let $\psi\colon W_{\Gamma_A} * W_{\Gamma_B} \to \mathbb{R}$ be any unbounded Aut-invariant quasimorphism which always exists according to Theorem 6.1. Then $\psi \circ p$ is an unbounded Aut^0-invariant quasimorphism on $W_{\Gamma'}$. Since the labelled graph automorphisms $\{\sigma_i\}$ form a set of coset representative for $\mathrm{Aut}^0(W_{\Gamma'})$ in $\mathrm{Aut}(W_{\Gamma'})$ according to Lemma 8.13 , the quasimorphism $\widehat{\psi \circ q}$ defined for $g \in W_{\Gamma'}$ by $\widehat{\psi \circ q}(g) = \sum_i \psi(q(\sigma_i(g)))$ is invariant under the whole automorphism group $\mathrm{Aut}(W_{\Gamma'})$ by Lemma 2.21. It remains to check that it is unbounded.

Any labelled graph automorphism $\sigma \in \mathrm{Aut}(W_{\Gamma'})$ satisfies by Lemma 9.19

$$\sigma(A) \cap A, \sigma(A) \cap B, \sigma(B) \cap A, \sigma(B) \cap A \in \{A, B\}.$$

Thus, any labelled graph automorphism either descends to an automorphism of $W_{\Gamma'_A} * W_{\Gamma'_B}$ or satisfies that $q(\sigma(w))$ is a single letter in $W_{\Gamma'_A} * W_{\Gamma'_B}$ for all words $w \in W_{\Gamma'_A} * W_{\Gamma'_B}$. Let $J \leq I$ be the subset of labelled graph automorphisms in $W_{\Gamma'}$ that descend to $W_{\Gamma'_A} * W_{\Gamma'_B}$. Then $|J| \geq 1$ and for all $g \in W_{\Gamma'_M} * W_{\Gamma'_N}$ considered as a subgroup of $W_{\Gamma'}$ one calculates

$$\widehat{\psi \circ q}(g) = \sum_i \psi(q(\sigma_i(g))) = |J| \psi(g).$$

Therefore, the Aut-invariant quasimorphism $\widehat{\psi \circ p}$ is unbounded on $W_{\Gamma'}$. Finally, linear independence of quasimorphisms constructed in this way follows from linear independence of the quasimorphisms in Theorem 6.1.

Second, consider the case where all non-trivial minimal equivalence classes A in W_a and B in W_b just consist of single vertices $A = \{v_1\}$ and $B = \{v_2\}$ with both vertex groups equal to $\mathbb{Z}/2$. Then we need to adjust our strategy from the previous part since D_∞ is bounded.

Since W_a and W_b are freely indecomposable, both of them are connected. Since it holds that $W_a * W_b \ncong D_\infty$, there exists a second vertex in at least one of them. So without loss of generality we assume there is $v_3 \in W_b$ that is adjacent to v_2. Replace W_a by just G_{v_1} since $\{v_1\}$ is a lower cone itself and denote the resulting projection map $W_{\Gamma'} \to G_{v_1} * W_b$ again by p for simplicity. Let σ be a labelled graph automorphism of $W_{\Gamma'}$. We will proceed similarly to the first case of the proof to show that the symmetrisation of the homogenisation of our counting quasimorphisms is unbounded, by showing that it is unbounded when we restrict to suitable subgroups of W_Γ. For this we distinguish the cases where G_{v_3} is not equal to $\mathbb{Z}/2$ and where it is.

If $G_{v_3} \neq \mathbb{Z}/2$ then $\sigma(v_3) = v_1$ and $\sigma(v_1) = v_3$ are impossible. So if we have $p(\sigma(v_1)) \neq 0$ and $p(\sigma(v_3)) \neq 0$, then either $\sigma(v_1), \sigma(v_3)$ both belong to W_b or $\sigma(v_1) = v_1$ and $\sigma(v_3) \in W_b$. Therefore, restricted to the subgroup $G_{v_1} * G_{v_3}$ either a labelled graph automorphism σ preserves the W_b-code or the result is a letter. Consequently, we calculate using Lemma 5.9 for the last equality that

$$(\widehat{\bar{f}_z^{W_b} \circ p})|_{G_{v_1} * G_{v_3}} = \sum_i \bar{f}_z^{W_b}(p(\sigma_i|_{G_{v_1} * G_{v_3}})) = |J| \cdot (\bar{f}_z^{W_b})|_{G_{v_1} * G_{v_3}} = |J| \cdot \bar{f}_z^{G_{v_3}},$$

where J is the subset of labelled graph automorphisms satisfying $\sigma(v_1)$ and $\sigma(v_3) \in W_b$. Since J contains the identity, $|J| \geq 1$. Therefore, $\widehat{\bar{f}_z^{W_b} \circ p}$ is unbounded by Proposition 4.11 which proves the result in this case.

However, if $G_{v_3} = \mathbb{Z}/2$, then $p(\sigma(G_{v_1} * (G_{v_2} \times G_{v_3}))) \leq D_\infty \leq W_a * W_b$ if $\sigma(v_2) = v_1$ or $\sigma(v_3) = v_1$ since v_2 and v_3 are adjacent. So $p(\sigma(G_{v_1} * (G_{v_2} \times G_{v_3})))$ is a subgroup of of a group belonging to the set $\{D_\infty, W_a, W_b\}$ unless $\sigma(v_1) = v_1$ and $\sigma(\{v_2, v_3\}) \subset W_b$. Similarly to the previous case we calculate

$$\begin{aligned}(\widehat{\bar{f}_z^{W_b} \circ p})|_{G_{v_1} * (G_{v_2} \times G_{v_3})} &= \sum_i \bar{f}_z^{W_b}(p(\sigma_i|_{G_{v_1} * (G_{v_2} \times G_{v_3})})) = |J| \cdot (\bar{f}_z^{W_b})|_{G_{v_1} * (G_{v_2} \times G_{v_3})} \\ &= |J| \cdot \bar{f}_z^{G_{v_2} \times G_{v_3}},\end{aligned}$$

where we again used Lemma 5.9 for the last equality and J denotes the subset of labelled graph automorphisms satisfying $\sigma(v_1) = v_1$ and $\sigma(\{v_2, v_3\}) \subset W_b$. As before, $|J| \geq 1$, so $\widehat{\bar{f}_z^{W_b} \circ p}$ is unbounded by Proposition 4.11. This concludes the last case and proves the result of the theorem. □

Corollary 9.28. *Let $\Gamma = (V, E)$ be a finite graph. Let $\{G_v\}_{v \in V}$ be a family of finite abelian groups and let $W(\Gamma, \{G_v\}_{v\in V})$ be their graph product. If no vertex group is equal to $\mathbb{Z}/2$ then either $W(\Gamma, \{G_v\}_{v\in V})$ is finite or $W(\Gamma, \{G_v\}_{v\in V})$ admits infinitely many linearly independent unbounded homogeneous Aut-invariant quasimorphisms.*

Proof. If the k-fold free product of groups of order two $Z_k = \mathbb{Z}/2 * \cdots * \mathbb{Z}/2$ is a truncated subgroup H of $W(\Gamma, \{G_v\}_{v\in V})$ on the vertex set $V' \subset V$, then for $v' \in V'$ all vertex groups $G_{v'}$ are groups of order two since H is the free product of the connected components of $\Gamma_{V'}$. Consequently, if there is $G_v \neq \mathbb{Z}/2$ for $v \in V$, then $W(\Gamma, \{G_v\}_{v\in V})$ does not decompose as a product of direct truncated subgroups each of which is isomorphic to some Z_k for $k \geq 2$. The result follows from Theorem 9.27. □

Remark 9.29. This is a super-strong version of the bq-dichotomy studied in [BGKM16], where instead of boundedness of a group G one has finiteness and instead of any unbounded quasimorphism on G one has unbounded quasimorphisms that are invariant under all automorphisms of G.

Theorem 9.30. $\Gamma = (V, E)$ *be a finite graph that is not complete and let W_Γ be a graph product of finite abelian groups on Γ. If there are no two vertices $v, w \in V$ such that $G_v = G_w = \mathbb{Z}/2$ and $lk(v) = lk(w)$, then W_Γ admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms.*

Proof. If the k-fold free product $Z_k = \mathbb{Z}/2 * \cdots * \mathbb{Z}/2$ is a direct truncated subgroup of W_Γ, then all vertices generating the free factors of Z_k have the same link. So by assumption Z_k can never be a direct truncated subgroup of W_Γ for any $k \geq 2$. Then the result follows from Theorem 9.27. □

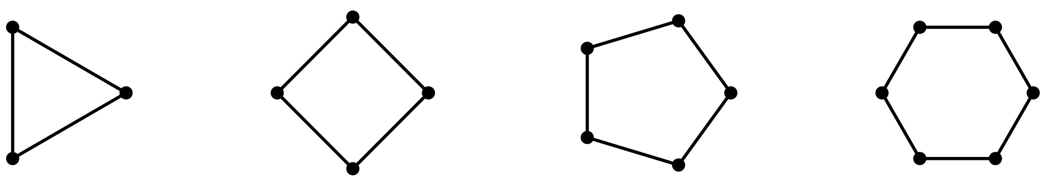
In particular, we obtain the following corollary as a special case.

Corollary 9.31. *Let $\Gamma = (V, E)$ be a finite graph in which no two vertices have the same link. Then either Γ is a complete graph in which case all graph products of finite groups on Γ are finite or any graph product of finite abelian groups on Γ admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms.*

10 Examples of graph products

In this section we consider some families of graphs as examples and analyse which graph products of finite groups and which right angled Artin groups on these graphs admit unbounded Aut-invariant quasimorphisms. For a graph Γ we will denote by W_Γ the graph product of a family of finite groups on Γ and by R_Γ the right angled Artin group on Γ. Implicitly, we expand Γ to Γ' so that $W_{\Gamma'} \cong W_\Gamma$ is a graph product where each vertex group is primary if this was not already the case for W_Γ. Further, recall that groups of the form $(D_\infty)^k \times A$ do not admit any unbounded Aut-invariant quasimorphisms for all $k \geq 0$ whenever A is abelian.

Example 10.1. Let Γ_n be the graph of the regular n-gon. If $n = 3$, then $W_{\Gamma_3} \cong G_{v_1} \times G_{v_2} \times G_{v_3}$ and $R_{\Gamma_3} \cong \mathbb{Z}^3$ are always abelian.

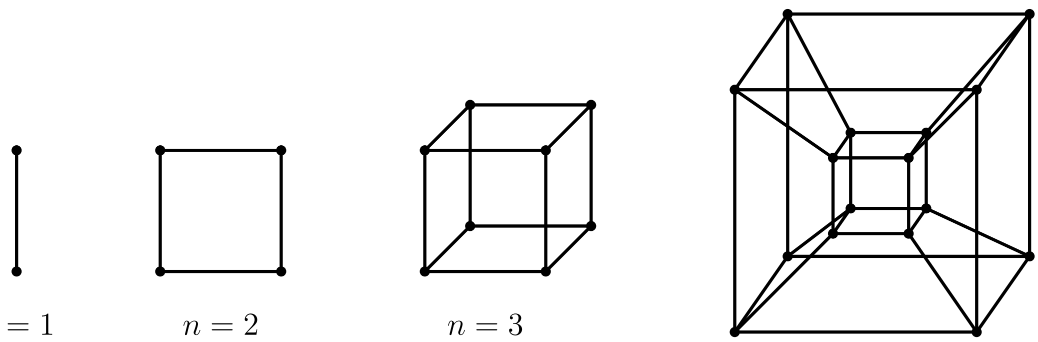


$n = 3$ $n = 4$ $n = 5$ $n = 6$

For $n \geq 5$ the links of no two vertices are equal and so W_{Γ_n} always admits infinitely many linearly independent Aut-invariant quasimorphisms according to Corollary 9.31. Moreover, the condition $lk(v) \subset st(w)$ is never satisfied for any distinct vertices v, w and so R_{Γ_n} does not admit any transvections. Therefore, it follows from Corollary 9.24 that R_{Γ_n} admits infinitely many linearly independent Aut-invariant quasimorphisms.

For $n = 4$ it follows from Theorem 9.27 that W_{Γ_4} has infinitely many linearly independent homogeneous Aut-invariant quasimorphisms except for the single case where all vertex groups are $\mathbb{Z}/2$ and so $W_{\Gamma_4} \cong D_\infty \times D_\infty$. Furthermore, for $R_{\Gamma_4} \cong F_2 \times F_2$ opposite vertices belong to the same equivalence class with respect to $\sim_\tau$. Both of these classes are minimal and it follows from Proposition 9.23 that R_{Γ_4} admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms.

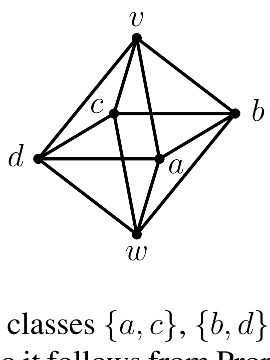
Example 10.2. Let Γ_n be the graph given by the 1-skeleton of the n-cube. Clearly, for $n = 1$ we have that $W_{\Gamma_1} \cong G_{v_1} \times G_{v_2}$ and $R_{\Gamma_1} \cong \mathbb{Z}^2$ are abelian. The case $n = 2$ is the case of the regular 4-gon discussed in Example 10.1 above.



$n = 1$ $n = 2$ $n = 3$ $n = 4$

Let $n \geq 3$. The links of no two vertices are equal and the condition $lk(v) \subset st(w)$ is never satisfied for any two distinct $v, w \in V_n$. Consequently, it follows from Corollary 9.31 and Corollary 9.24 that W_{Γ_n} and R_{Γ_n} always admit infinitely many linearly independent Aut-invariant quasimorphisms.

Example 10.3 (Platonic solids)**.** The graph given by the tetrahedron T is the complete graph on 4 vertices and therefore $W_T \cong G_{v_1} \times G_{v_2} \times G_{v_3} \times G_{v_4}$ and $R_T \cong \mathbb{Z}^4$ are abelian. The cube C is the case $n = 3$ of Example 10.2 above. For the icosahedron I and the dodecahedron D the conditions $lk(v) = lk(w)$ and $lk(v) \subset st(w)$ are both never satisfied for distinct vertices v, w. It follows from Corollary 9.31 and Corollary 9.24 that W_I, W_D, R_I and R_D all admit infinitely many linearly independent Aut-invariant quasimorphisms. Let O be the graph of the octahedron pictured below.



In R_O there are the three equivalence classes $\{a, c\}$, $\{b, d\}$ and $\{v, w\}$, all of which are minimal with respect to $\sim_\tau$ and generate F_2. So it follows from Proposition 9.23 that R_O admits infinitely many linearly independent Aut-invariant quasimorphisms. In W_O any two opposite vertices have the same link and $W_O \cong (G_v * G_w) \times (G_a * G_c) \times (G_b * G_d)$ as a product of direct truncated subgroups. It follows from Theorem 9.27 that W_O admits infinitely many linearly independent Aut-invariant quasimorphisms unless all vertex groups are equal to $\mathbb{Z}/2$ in which case W_O is isomorphic to $(D_\infty)^3$.

In the above examples we can never find lower cones invariant under labelled graph automorphisms that are not the whole graph. Let us see in some less symmetrical graphs how we can extract Aut-invariant quasimorphisms by more elementary means from lower cones and Proposition 9.21 without appealing to the full strength of Theorem 9.27.

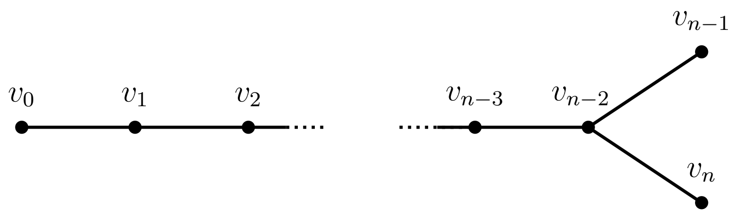
Example 10.4. Let A_n denote the graph given by the standard triangulation of the interval $[0, n]$ with all integers in $[0, n]$ as vertices and edges of length one in between them.

For $n = 0, 1$ the groups W_{A_n} and R_{A_n} are all abelian and so none of them admit unbounded Aut-invariant quasimorphisms.

For $n \geq 2$ the set $\{v_0, v_n\}$ is a lower cone that is invariant under labelled graph automorphisms. Then it follows directly from Proposition 9.21 that R_{A_n} and W_{A_n} admit infinitely many linearly independent homogeneous Aut-invariant quasimorphisms except for W_{A_n} in the case where $G_{v_0} = G_{v_n} = \mathbb{Z}/2$. So assume that $G_{v_0} = G_{v_n} = \mathbb{Z}/2$ for the rest of this example. For $n \geq 4$ the set $\{v_0, v_1, v_{n-1}, v_n\}$ forms a lower cone as well and Proposition 9.21 again yields the existence of infinitely many linearly independent homogeneous Aut-invariant quasimorphisms. For $n = 2$ it holds that $W_{A_2} \cong G_{v_1} \times D_\infty$ and so W_{A_2} does not admit any unbounded quasimorphisms since D_∞ does not according to Example 2.13 and G_{v_2} is finite. Only to settle the case $n = 3$ we note that no two distinct vertices have the same link and so Corollary 9.31 yields the existence of infinitely many linearly independent Aut-invariant quasimorphisms.

Remark 10.5. The particular choice of lower cone is usually not unique. For example, for $n \geq 8$ the set $\{v_2, v_3, v_5, v_6\}$ is a lower cone that intersects trivially with the lower cone $\{v_0, v_1, v_{n-1}, v_n\}$ chosen in the argument before. Different choices of lower cones yield entirely different Aut-invariant quasimorphisms.

Example 10.6. Let B_n denote the graph which is A_{n-2} with two additional vertices attached to the vertex $n - 2$. So B_n has $n + 1$ vertices. Then B_2 is the graph A_2 of Example 10.4 above.



For $n \geq 4$ the set $\{v_{n-2}, v_{n-1}, v_n\}$ forms a lower cone of W_{B_n} and R_{B_n} respectively. Moreover, $\{v_0\}$ is a minimal equivalence class with respect to $\sim_\tau$. So $\{v_0, v_{n-2}, v_{n-1}, v_n\}$ forms a lower cone L of W_{B_n} and R_{B_n}. L is invariant under labelled graph automorphisms and the truncated subgroup generated by L is isomorphic to $G_{v_0} * \big(G_{v_{n-2}} \times (G_{v_{n-1}} * G_{v_n})\big)$. Since the group $G_{v_{n-2}} \times (G_{v_{n-1}} * G_{v_n})$ has non-trivial center, it is freely indecomposable. So Proposition 9.21 implies that W_{B_n} and R_{B_n} admit infinitely many linearly independent homogeneous Aut-invariant quasimorphisms for all $n \geq 4$.

For $n = 3$ the vertices $\{v_0, v_2, v_3\}$ form a lower cone. By Proposition 9.21 the graph product $W_{B_3} \cong G_{v_1} \times (G_{v_0} * G_{v_2} * G_{v_3})$ admits infinitely many linearly independent quasimorphisms if $G_{v_0}, G_{v_2}, G_{v_3}$ are not all equal to $\mathbb{Z}/2$.

Remark 10.7. With our methods we cannot settle the case $n = 3$ for B_n fully since we do not know whether F_3 or $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ admit unbounded Aut-invariant quasimorphisms.

11 Aut-invariant stable commutator length

For any group G let cl_G denote the commutator length on $[G,G]$, which is defined to be the minimal number of commutators required to produce a given element of the commutator subgroup. Let $\mathrm{scl}_G(x) = \lim_n \frac{\mathrm{cl}(x^n)}{n}$ denote the stable commutator length of $x \in [G,G]$. It shares a deep relationship with quasimorphisms on G through the so-called Bavard duality [Bav91]. We will now define the Aut-invariant (stable) commutator length; this is a special case of the so-called $\hat{G}$-invariant (stable) commutator length defined in [KaKi20] for groups $\hat{G}$ in which G is a normal subgroup.

Definition 11.1. Let $G \leq \hat{G}$ be a normal subgroup. Consider the subgroup $[\hat{G}, G] \leq G$ generated by commutators of the form $[F, g]$ and their inverses where $F \in \hat{G}$ and $g \in G$. Then for $x \in [\hat{G}, G]$ the *$\hat{G}$-invariant commutator length* $\mathrm{cl}_{\hat{G},G}(x)$ is defined to be the minimal length of an expression of x as a product of commutators $[F, g]$ and their inverses where $F \in \hat{G}$ and $g \in G$. The *$\hat{G}$-invariant stable commutator length* $\mathrm{scl}_{\hat{G},G}$ is defined for $x \in [\hat{G}, G]$ by $\mathrm{scl}_{\hat{G},G}(x) = \lim_n \frac{\mathrm{cl}_{\hat{G}G}(x^n)}{n}$.

Given any group G, its inner automorphism group $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and so the above definition applies to $\mathrm{Inn}(G)$. If G has trivial center, G can be identified with $\mathrm{Inn}(G)$. In this case we simplify the notation by denoting the *Aut(G)-invariant commutator length* simply as $\mathrm{cl}_{\mathrm{Aut}}$ and the *Aut(G)-invariant stable commutator length* simply as $\mathrm{scl}_{\mathrm{Aut}}$.

Setting $\hat{G} = \mathrm{Aut}(G)$ the following lemma is proven in [KaKi20, Lemma 2.1].

Lemma 11.2. *Let G be a group with trivial center so that $G = \mathrm{Inn}(G)$. Let ϕ be a homogeneous Aut-invariant quasimorphism on G. Then any $x \in [\mathrm{Aut}(G), G] \leq G$ satisfies*

$$\mathrm{scl}_{\mathrm{Aut}}(x) \geq \frac{1}{2}\frac{|\phi(x)|}{D(\phi)}.$$

□

In fact, according to [KaKi20, Theorem 1.3] $\hat{G}$-invariant quasimorphisms satisfy an analogue Bavard's duality theorem if $[\hat{G}, G] = G$. All free products $A * B$ of freely indecomposable groups A and B have trivial center and so the notions $\mathrm{cl}_{\mathrm{Aut}}$ and $\mathrm{scl}_{\mathrm{Aut}}$ apply. However, free products often fail to satisfy $[\mathrm{Aut}(G), G] = G$. We will use a constructive approach rather than relying on an invariant analogue of Bavard's duality in the following.

Example 11.3. For $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ it holds that $\mathrm{scl}_{\mathrm{Aut}} \equiv 0$. To see this denote the generators of the $\mathbb{Z}/2$ factors by a and b and let s be the automorphism of D_∞ interchanging a and b. Then we calculate $[s, a] = sas^{-1}a^{-1} = s(a)a^{-1} = ba$ and $[s, b] = ab = (ba)^{-1}$. Since the total number of letters appearing in any expression of the form $[f, x]$ is always even where $f \in \mathrm{Aut}(D_\infty)$ and $x \in D_\infty$, it holds that $[\mathrm{Aut}(D_\infty), D_\infty)] \cong \mathbb{Z}$ generated by ba. In fact, any power $(ba)^k$ for $k \in \mathbb{Z}$ can be written as a single commutator $[s, w]$, where w is one of the two words of length $|k|$. Thus, $\mathrm{cl}_{\mathrm{Aut}}$ is equal to one for any non-trivial element in $[\mathrm{Aut}(D_\infty), D_\infty)]$ and $\mathrm{scl}_{\mathrm{Aut}}$ vanishes.

Example 11.4. Consider $G = \mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}/3 * \mathbb{Z}/2$. Then $\mathrm{Aut}(G)$ is generated by the set C consisting of the non-trivial factor automorphism of $\mathbb{Z}/3$ and conjugations by letters of $\mathbb{Z}/3$ and $\mathbb{Z}/2$, since both free factors are abelian groups. Consequently, $[\mathrm{Aut}(G), G]$ is normally generated by commutators of the form $[c, g] = cgc^{-1}g^{-1} = c(g)g^{-1}$ for $c \in C$ and $g \in G$. In all expressions $[c, g]$ the letter b representing the non-trivial element of the factor $\mathbb{Z}/2$ arises an even number of times. Therefore, $b \notin [\mathrm{Aut}(G), G]$ and the latter is not the full group G.

11.1 In free products

Lemma 11.5. *Let $G = A * B$ be a free product of freely indecomposable groups where at least one of the factors is infinite cyclic. Then $[\mathrm{Aut}(G), G]$ has index at most 2 in G. Therefore, any unbounded quasimorphism on G is unbounded when restricted to $[\mathrm{Aut}(G), G]$.*

Proof. If a quasimorphism $q\colon G \to \mathbb{R}$ is bounded by a constant C on a finite index subgroup $H \leq G$, then the image of q is bounded on G by $C + D_q + \max_i |q(g_i)|$ where $\{g_i\}$ is a finite system of coset representatives of H in G. Therefore, any quasimorphism q with unbounded image cannot be bounded on H. So, it remains to show that the index of $[\mathrm{Aut}(G), G]$ in G is finite to prove the lemma.

First, consider the case where A and B are both infinite cyclic and so G can be identified with F_2, the free group of rank 2. Let x and y be standard generators. Consider the automorphism φ of F_2 defined by $\varphi(x) = yx$ and $\varphi(y) = y$. Then $[\varphi, x] = \varphi(x) \cdot x^{-1} = y$ and thus $\langle y\rangle \leq [\mathrm{Aut}(F_2), F_2]$. By symmetry of the generating set it holds that $\langle x\rangle \leq [\mathrm{Aut}(F_2), F_2]$ as well and it follows that $[\mathrm{Aut}(F_2), F_2] = F_2$.

Second, consider the case where only one of the factors is infinite cyclic. Without loss of generality assume $A = \mathbb{Z}$. Let x denote a generator of A. For any $b \in B$ we can define the transvection φ_b on x by $\varphi_b(x) = bx$ and by $\varphi_b(b') = b'$ for all $b' \in B$. Then φ_b is an automorphism and satisfies $[\varphi_b, x] = \varphi_b(x) \cdot x^{-1} = b$ for all $b \in B$. Thus, $B \leq [\mathrm{Aut}(G), G]$. Denote the non-trivial factor automorphism of $A = \mathbb{Z}$ by f. Then $[f, x] = f(x) \cdot x^{-1} = x^{-2}$ and we deduce that $2\mathbb{Z} \leq [\mathrm{Aut}(G), G]$. Therefore, $2\mathbb{Z} * B \leq [\mathrm{Aut}(G), G]$. Since $[\mathrm{Aut}(G), G]$ is a normal subgroup, it holds that $N \leq [\mathrm{Aut}(G), G]$ where N is the normal closure of $2\mathbb{Z} * B$. However, $G/N \leq \mathbb{Z}/2$ and so $[\mathrm{Aut}(G), G]$ has at most index 2 in G. □

Theorem 11.6. *Let $G = A * B$ be a free product of freely indecomposable groups and assume that G is not the infinite dihedral group. Then there always exist elements $g \in G$ with positive Aut-invariant stable commutator length $\mathrm{scl}_{\mathrm{Aut}}(g) > 0$.*

Proof. If one of the factors is infinite cyclic, Theorem 6.1 implies the existence of an unbounded Aut(G)-invariant homogeneous quasimorphism, which is unbounded on $[\mathrm{Aut}(G), G]$ according to Lemma 11.5. The statement then follows from Lemma 11.2.

Assume from now on that neither A nor B is infinite cyclic. We will prove the theorem by explicitly constructing an element in $[\mathrm{Aut}(G), G]$ together with a homogeneous Aut-invariant quasimorphism which is non-trivial on that element. The only non-trivial group with no non-trivial automorphisms is $\mathbb{Z}/2$. Thus, one of the factors has to have a non-trivial automorphism since $A * B$ is not the infinite dihedral group.

Without loss of generality assume that A satisfies $|\mathrm{Aut}(A)| \geq 2$. Let $a_1 \in A$ be such that $f(a_1) = a_2 \neq a_1$ for some $f \in \mathrm{Aut}(A)$. Then we compute that $[f, a_1] = a_2a_1^{-1}$ is a non-trivial element in $[\mathrm{Aut}(G), G]$. Since $[\mathrm{Aut}(G), G]$ is normal, it holds for any $h \in B$ that $ha_2a_1^{-1}h^{-1} \in [\mathrm{Aut}(G), G]$. Thus, $(a_2a_1^{-1}ha_2a_1^{-1}h^{-1})^k \in [\mathrm{Aut}(G), G]$ for any $k \in \mathbb{N}$ and $h \in B$.

First, assume that A and B are not isomorphic. Since $|\operatorname{Aut}(A)| \geq 2$, it holds that $|A| \geq 3$ and we can choose a non-trivial $a \neq a_2 a_1^{-1}$ and fix some non-trivial $h \in B$. We define for $n_1, \ldots, n_\ell \in \mathbb{N}$ the word

$$w = \prod_{i=1}^{\ell} \left([a,h](a_2 a_1^{-1} h a_2 a_1^{-1} h^{-1})^{n_i}\right) \in [\operatorname{Aut}(G), G].$$

We calculate

$$A\text{-code}(w) = \begin{cases} (1,1,2n_1,1,1,2n_2,\ldots,1,1,2n_\ell) & \text{if } a^{-1} \neq a_2 a_1^{-1}, \\ (1,2n_1+1,1,2n_2+1,\ldots,1,2n_\ell+1) & \text{if } a^{-1} = a_2 a_1^{-1}. \end{cases}$$

Set $z = A\text{-code}(w)$. For all $n \in \mathbb{N}$ it holds that $A\text{-code}(w^n) = (z, \ldots, z)$. Choose $\ell \geq 3$ together with large and distinct $n_1, \ldots, n_\ell \in \mathbb{N}$, which implies that z is generic. It follows from Proposition 4.11 that $\bar{f}_z^A$ is an Aut-invariant quasimorphism. By construction f_z^A satisfies $f_z^A(w^n) = n$ for all $n \in \mathbb{N}$ and so $\bar{f}_z^A(w) > 0$ for our choice of $w \in [\operatorname{Aut}(G), G]$. The statement then follows from Lemma 11.2.

Second, assume $A \cong B$. Let s denote a swap automorphism. Set $b_i = s(a_i)$ for $i \in \{1,2\}$. Then the element $[sfs^{-1}, b_1] = b_2 b_1^{-1} \in [\operatorname{Aut}(G), G]$ is non-trivial and belongs to the factor B. Set $b = s(a)$. Then $a^{-1} = a_2 a_1^{-1}$ is equivalent to $b^{-1} = b_2 b_1^{-1}$. For $n_1, \ldots, n_\ell \in \mathbb{N}$ we define the word

$$w = \prod_{i=1}^{\ell} \left([a,b](a_2 a_1^{-1} b_2 b_1^{-1})^{n_i}\right) \in [\operatorname{Aut}(G), G].$$

Observe, that $A\text{-code}(w) = B\text{-code}(w)$. As in the previous case,

$$A\text{-code}(w) = \begin{cases} (1,1,n_1,1,1,n_2,\ldots,1,1,n_\ell) & \text{if } a^{-1} \neq a_2 a_1^{-1}, \\ (1,n_1+1,1,n_2+1,\ldots,1,n_\ell+1) & \text{if } a^{-1} = a_2 a_1^{-1}. \end{cases}$$

Set $z = A\text{-code}(w)$ as before and choose $\ell \geq 3$ and $n_1, \ldots, n_\ell \in \mathbb{N}$ large enough and distinct, so that z is generic. Again, we calculate $f_z^A(w^n) = n = f_z^B(w^n)$, which implies that $(\bar{f}_z^A + \bar{f}_z^B)(w) > 0$. Proposition 4.11 implies that $\bar{f}_z^A + \bar{f}_z^B$ is an homogeneous Aut-invariant quasimorphism and so applying Lemma 11.2 concludes the proof. □

Corollary 11.7. *Let $G = G_1 * \cdots * G_k$ be a free product of freely indecomposable groups. Assume that there exist distinct $i, j \in \{1, \ldots, k\}$ such that G_i, G_j are not both equal to $\mathbb{Z}/2$ and any other free factor G_k neither is isomorphic to G_i or G_j nor is infinite cyclic. Then there exist $g \in G$ with positive Aut-invariant stable commutator length* $\operatorname{scl}_{\mathrm{Aut}}(g) > 0$.

Proof. Let $H = G_i * G_j$. We claim that the projection $p \colon G \to H$ is Aut-equivariant, i.e. any automorphism of G descends via p to an automorphism of $G_j * G_k$. This is equivalent to $\ker(p)$ being a characteristic subgroup of G.

Recall, that by [Gil87] $\operatorname{Aut}(G)$ is generated by factor automorphisms, swap automorphisms, partial conjugations and transvections. It is clear that all factor automorphisms and all partial conjugations of G descend to automorphisms of $G_j * G_k$ via p. By our assumption there are no swap automorphisms permuting any other free factors in G with G_j and G_k, so these descend to the quotient as well. It only remains to check the equivariance of the projection with respect to transvections for the case where G_j or G_k happen to be infinite cyclic. So let G_j be infinite cyclic generated by x and let a be a letter from a different factor G_ℓ. If $\ell = k$, then any transvection φ_a defined by $\varphi_a(x) = ax$ or $\varphi_a(x) = xa$ descends via p to the same transvection on $G_j * G_k$. If $\ell \neq k$, any such transvection descends to the identity on $G_j * G_k$. Since a generating set of $\mathrm{Aut}(G)$ descends to automorphisms of the quotient $G_j * G_k$, any element of $\mathrm{Aut}(G)$ does so. Consequently, the map p is Aut-equivariant. In fact, p induces a surjection $\mathrm{Aut}(G) \to \mathrm{Aut}(H)$.

Therefore, p induces a surjective map $p_\star \colon [\mathrm{Aut}(G), G] \to [\mathrm{Aut}(H), H]$. Since all $g \in G$ satisfy $\mathrm{scl}_{\mathrm{Aut}}(p(g)) \leq \mathrm{scl}_{\mathrm{Aut}}(g)$, the result follows by applying Theorem 11.6 to H. □

We will now give a few more examples of free products G where $[\mathrm{Aut}(G), G] = G$ and therefore the notions of Aut-invariant (stable) commutator length are defined on all elements of G.

Example 11.8. For a product $G = A * B$ of two freely indecomposable perfect groups A and B it holds that $[\mathrm{Aut}(G), G] = G$. Indeed, since A perfect, it holds that $A = [A, A] \leq [\mathrm{Aut}(G), G]$. Similarly, $B \leq [\mathrm{Aut}(G), G]$ and A and B generate G it follows that $[\mathrm{Aut}(G), G] = G$.

Example 11.9. $G = \mathbb{Z}/p * \mathbb{Z}/q$ satisfies $[\mathrm{Aut}(G), G] = G$ for $p, q \geq 3$ prime. Indeed, let $m \colon \mathbb{Z}/p \to \mathbb{Z}/p$ be multiplication by 2, which is a factor automorphism. Consider the standard generator $1_p \in \mathbb{Z}/p$. It holds that $[m, 1_p] = m(1_p) - 1_p = 1_p$. Thus, $\mathbb{Z}/p \leq [\mathrm{Aut}(G), G]$. Similarly, $\mathbb{Z}/q \leq [\mathrm{Aut}(G), G]$ and so $[\mathrm{Aut}(G), G] = G$.

Example 11.10. Let $k, \ell \geq 2$. Then $G = A^k * B^\ell$ satisfies $[\mathrm{Aut}(G), G] = G$ for all non-trivial abelian groups A and B. For simplicity of notation consider the case $k = 2$. Let ϕ be the factor automorphism defined by $\phi(a, 0) = (a, a)$ and $\phi(0, a) = (0, a)$ for all $a \in A$. Then $[\phi, (a, 0)] = \phi(a, 0) - (a, 0) = (a, 0)$ for all $a \in A$. Thus, $A \times 0 \leq [\mathrm{Aut}(G), G]$. Analogously, $0 \times A \leq [\mathrm{Aut}(G), G]$. Since these factors generate A^2 it follows that $A^2 \leq [\mathrm{Aut}(G), G]$. Similarly, $B^\ell \leq [\mathrm{Aut}(G), G]$ and so $[\mathrm{Aut}(G), G] = G$.

Example 11.11. In all of the above examples for $G = A * B$ the equality $[\mathrm{Aut}(G), G] = G$ is always derived by showing that both factors A and B form subgroups of $[\mathrm{Aut}(G), G]$. Thus, any combination of free factors appearing in the three examples above still satisfies this equality, for example $A = \mathbb{Z}/p$ for $p \geq 3$ prime and B any freely indecomposable perfect group.

11.2 In graph products

For many graph products of finitely generated abelian groups the invariant $\mathrm{scl}_{\mathrm{Aut}}$ is defined on a subgroup of finite index. In fact, it is often defined on the whole graph product.

Proposition 11.12. *Let $\Gamma = (V, E)$ be a graph. Let $\{G_v\}_{v \in V}$ be a collection of finitely generated abelian groups and let $G = W(\Gamma, \{G_v\}_{v \in V})$ be their graph product. Assume that no vertex v satisfying $st(v) = V$ has a vertex group satisfying $|G_v| = \infty$. Let $\hat{G} = \mathrm{Aut}(G)$. Then $[\hat{G}, G]$ is a subgroup of finite index in G. Moreover, if no vertex $v \in V$ satisfies $st(v) = V$ and no element inside any vertex group G_v has infinite order or order equal to a power of 2, then $[\hat{G}, G] = G$.*

Proof. Since $W(\Gamma, \{G_v\}_{v \in V})$ is isomorphic to a graph product of groups where every vertex group is either primary or infinite cyclic, we assume without loss of generality that this is the case for $\Gamma = (V, E)$ and $\{G_v\}_{v \in V}$ and write $W_\Gamma = W(\Gamma, \{G_v\}_{v \in V})$. Moreover, if no element inside any vertex group has infinite order or order equal to a power of 2, then all vertex groups can be assumed to be primary for primes $\neq 2$.

By Corollary 8.10 the center of W_Γ is generated by all $v \in V$ satisfying $st(v) = V$. Since all vertex groups G_v of those vertices are finite abelian groups, the center of W_Γ is a finite abelian group and we may assume up to passing to a finite index subgroup of W_Γ, that W_Γ has trivial center.

The abelianisation $W_\Gamma/[W_\Gamma, W_\Gamma]$ is obtained by taking the quotient over all $[G_v, G_w]$ for $v, w \in V$. Consequently, $W_\Gamma/[W_\Gamma, W_\Gamma] = \bigoplus_{v \in V} G_v$. If no vertex group G_v has infinite order this implies that $[\hat{W_\Gamma}, W_\Gamma]$ has finite index, since $[W_\Gamma, W_\Gamma] \leq [\hat{W_\Gamma}, W_\Gamma]$. Otherwise let $v \in V$ be such that G_v is infinite cyclic and by abuse of notation denote a generator of G_v by v as well. Since $st(v) \neq V$, conjugation by v defines a nontrivial element inside $\mathrm{Aut}(W_\Gamma)$. Let $r \in \mathrm{Aut}(W_\Gamma)$ be the factor automorphism of W_Γ that inverts v. In $\mathrm{Aut}(W_\Gamma)$ we compute $[r, v^{-1}] = r(v^{-1}) \cdot v = v^2$. Thus, the subgroup generated by v^2 is contained in $[\hat{W_\Gamma}, W_\Gamma]$ as well. Let H be the group $\bigoplus_{v \in V'} G_v \oplus \bigoplus_{v \in V \setminus V'} G_v/2$ where $V' = \{v \in V : \ |G_v| < \infty\}$. H is a finite group and maps surjectively onto $W_\Gamma/[\hat{W_\Gamma}, W_\Gamma]$, which implies that $[\hat{W_\Gamma}, W_\Gamma]$ has finite index in W_Γ.

For the second part of the statement assume that all vertex groups are primary for primes not equal to 2. Let $k \geq 1$ and let $p \neq 2$ be a prime. Then $m\colon \mathbb{Z}/p^k \to \mathbb{Z}/p^k$ induced by multiplication by 2 is an automorphism. Consider $G_v = \mathbb{Z}/p^k$ and regard m as the corresponding factor automorphism of W_Γ. Then $[m, v] = m(v) \cdot v^{-1} = v^2 \cdot v^{-1} = v$. Therefore, G_v is a subgroup of $[\hat{W_\Gamma}, W_\Gamma]$. Since W_Γ is generated by the vertex groups, it follows that $[\hat{W_\Gamma}, W_\Gamma] = W_\Gamma$. □

Let Z_k be the k-fold free product of groups of order two $Z_k = \mathbb{Z}/2 * \cdots * \mathbb{Z}/2$.

Proposition 11.13. *Let $\Gamma = (V, E)$ be a finite graph in which $st(v) \neq V$ for all $v \in V$. Then there always exist elements of positive Aut-invariant stable commutator length in the following groups:*

1. *$W(\Gamma, \{G_v\}_{v \in V})$, the graph product of a family of finite abelian groups $\{G_v\}_{v \in V}$, if the group $W(\Gamma, \{G_v\}_{v \in V})$ does not decompose as a product $G_1 \times \cdots \times G_\ell$ for $\ell \geq 1$ where each G_i is a direct truncated subgroup that is isomorphic to some Z_k for $k \geq 2$ or finite abelian.*

2. *R_Γ, the right angled Artin group on Γ, if one of the following two conditions is satisfied:*

   - *There is a minimal equivalence class M of $\sim_\tau$ such that $R_{\Gamma_M} \cong F_2$,*
   - *No equivalence class N of $\sim_\tau$ satisfies $R_{\Gamma_N} \cong F_k$ for $k \geq 2$.*

Proof. By assumption all groups have trivial center according to Corollary 8.10 Moreover, $W(\Gamma, \{G_v\}_{v \in V})$ is not finite since it contains a non-trivial free product. There exist unbounded homogeneous Aut-invariant quasimorphisms on $W_\Gamma = W(\Gamma, \{G_v\}_{v \in V})$ and R_Γ according to Theorem 9.27 and Proposition 9.23. Since $[\widehat{W_\Gamma}, W_\Gamma] \leq W_\Gamma$ and $[\widehat{R_\Gamma}, R_\Gamma] \leq R_\Gamma$ have finite index by Proposition 11.12 these quasimorphisms are unbounded on $[\widehat{W_\Gamma}, W_\Gamma]$ and $[\widehat{R_\Gamma}, R_\Gamma]$ respectively. Therefore, Lemma 11.2 implies the result. □

12 Open questions

There are various open questions that could be investigated further. The most immediate ones are whether $Z_k = \mathbb{Z}/2 * \cdots * \mathbb{Z}/2$ and the free group F_k admit unbounded Aut-invariant quasimorphisms for any $k \geq 3$.

It has been shown in [BrMa19] that F_k does admit unbounded Aut-invariant norms for any $k \geq 3$, which is a necessary condition for the existence of unbounded Aut-invariant quasimorphisms by Lemma 2.9. However, despite extensive studies of $\mathrm{Aut}(F_k)$ it remains unknown whether any free group of rank $k \geq 3$ admits an unbounded Aut-invariant quasimorphism [BrMa19, Remark 5.2]. Brandenbursky and Marcinkowski construct their Aut-invariant quasimorphisms on F_2 using the action of the mapping class group of punctured surface that only happens to have finite index in $\mathrm{Aut}(F_{2g})$ for $g = 1$. Note that the case of free groups of higher rank is complicated further by the fact that the projection maps $F_k \to F_{k-1}$ are not invariant under transvections.

For the case of Z_k where $k \geq 3$ note that proving the existence of an unbounded homogeneous Aut-invariant quasimorphism ψ for Z_k and proving the non-existence of unbounded Aut-invariant quasimorphisms for Z_{k-1} implies the existence of unbounded Aut-invariant quasimorphisms on Z_m for all $m \geq k$. In fact, the averaging procedure outlined in Lemma 2.21 applied to $\psi \circ p$ where p denotes the projection $Z_m \to Z_k$ onto the first k free factors will produce an Aut-invariant quasimorphism on Z_k. If every Aut-invariant quasimorphism on Z_{k-1} is bounded, then $\widehat{\psi \circ p}$ will restrict to a linear multiple of ψ on the first k-factors which proving unboundedness.

Finally, in the realm of graph products it is natural to ask questions about the existence of unbounded Aut-invariant quasimorphisms on graph products W for more general vertex groups or for underlying graphs with infinite vertex sets. However, a general description of the automorphism group of such graph products similar to the one given in in Section 8.2 from [CoGu09] for finitely generated abelian vertex groups and finite graphs is not known to the author of this thesis. Nevertheless, since the description of the automorphism group of free products coming from [FoRa40] and [FoRa41] holds for all freely indecomposable factors, it seems very plausible that our arguments apply to a larger class of graph products since the Aut-invariant quasimorphisms we construct arise from Aut^0-invariant projections to free products.

In a larger context the question on the cup product structure in the second bounded cohomology of free products has recently gained attention. It is conjectured that the cup product map $\smile\colon H_b^2(F_2,\mathbb{R}) \times H_b^k(F_2,\mathbb{R}) \to H_b^{k+2}(F_2,\mathbb{R})$ is trivial for all k. In the case where $k = 2$ the above conjecture is known to be true for coboundaries of Brooks quasimorphisms by [BuMo18] and Rolli quasimorphisms by [Heu17], in which Heuer makes use of a specific decomposability condition that is called Δ-decomposability. Furthermore, it is known for certain infinite sums of Brooks quasimorphisms by Francesco Fournier-Facio [Fou20]. A stronger triviality result for Δ-decomposable quasimorphisms concerning higher k has recently been proven in [AmBu21]. Currently, to the best knowledge of the author nothing is known for the cup product of coboundaries of code quasimorphisms and little is known about the cup product structure on general free products or graph products. Thus, it seems desirable to the author to attempt explicit computations regarding code quasimorphisms and their pullbacks to graph products.

Appendix A:
Aut-invariant Brooks quasimorphisms on free products

In this appendix we show that for a free product of two finite non-isomorphic groups unbounded Aut-invariant quasimorphisms can be constructed purely from Brooks counting quasimorphisms without any additional constructions of codes.

Let A and B be two non-isomorphic finite groups and let $G = A * B$. By Lemma 3.4 $\mathrm{Out}(G)$ is generated by the images of $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ acting as factor automorphisms. Since

$\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ commute in $\mathrm{Aut}(G)$ we have that a set of representatives of $\mathrm{Out}(G)$ is given by products of factor automorphisms of A and B. Thus, $|\mathrm{Out}(G)| \le |\mathrm{Aut}(A) \times \mathrm{Aut}(B)|$ is finite. Let $w \in G$ be a cyclically reduced word and denote by $f_w^{Br} \colon G \to \mathbb{R}$ Brooks's counting quasimorphism that counts the difference of disjoint occurrences of w and disjoint occurrences of w^{-1} inside any word. It is clear that for any factor automorphism $\varphi \in \mathrm{Aut}(G)$ the equality

$$f_{\varphi(w)}^{Br}(\varphi(g)) = f_w^{Br}(g)$$

holds for all $g \in G$. Consequently, all automorphisms $\varphi_0 \in \mathrm{Aut}(A) \times \mathrm{Aut}(B) \subset \mathrm{Aut}(G)$ satisfy $f_{\varphi_0(w)}^{Br}(g) = f_w^{Br}(\varphi_0^{-1}(g))$ for all $g \in G$. Define $\psi_w^{Br} \colon G \to \mathbb{R}$ by

$$\psi_w^{Br} = \sum_{\varphi \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)} f_{\varphi(w)}^{Br},$$

where the finiteness of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ is crucial. Then ψ_w^{Br} is a finite sum of Brooks counting quasimorphisms on G. Moreover, for $\theta \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ we calculate for $g \in G$ that

$$\begin{aligned}
\psi_w^{Br}(\theta(g)) &= \sum_{\varphi \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)} f_{\varphi(w)}^{Br}(\theta(g)) \\
&= \sum_{\varphi \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)} f_{(\theta^{-1} \circ \varphi)(w)}^{Br}(g) \\
&= \sum_{\varphi' \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)} f_{\varphi'(w)}^{Br}(g) \\
&= \psi_w^{Br}(g).
\end{aligned}$$

Lemma 12.1. *Let w be a cyclically reduced word and let A, B be two non-isomorphic non-trivial finite groups. Then the homogenisation $\bar{\psi}_w^{Br}$ of ψ_w^{Br} is an Aut-invariant quasimorphism on $A * B$ that is a finite sum of homogenisations of Brooks counting quasimorphisms.*

Proof. The calculation above proves invariance of ψ_w^{Br} under $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ which contains a set of representatives of $\mathrm{Out}(A * B)$ according to Lemma 3.4. So by Lemma 2.19 the quasimorphism $\bar{\psi}_w^{Br}$ is invariant under all automorphisms of $A * B$. ☐

It remains to verify that not all quasimorphisms of the form $\bar{\psi}_w^{Br}$ are bounded. If A and B are not isomorphic and both non-trivial at least one of them has cardinality ≥ 3. Let us assume that $|A| \geq 3$ and choose distinct non-trivial $a_0, a_1 \in A$. Let $b \in B$ be non-trivial and consider the word

$$x = a_0 b a_1 b a_1 b a_0 b a_0 b a_0 b a_1 b a_1 b a_1 b a_1 b.$$

Proposition 12.2. *Let A, B be two non-isomorphic non-trivial finite groups where $|A| \geq 3$. Then $\bar{\psi}_x^{Br}$ is an unbounded Aut-invariant quasimorphism on $A * B$ that is a finite sum of homogenisations of Brooks's counting quasimorphisms for any choice of x as above.*

Proof. Let $\varphi \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ and let $\varphi(a_i) = a_i' \in A$ for $i \in \{1, 2\}$ and $\varphi(b) = b' \in B$. Then we have

$$\varphi(x) = a_0' b' a_1' b' a_1' b' a_0' b' a_0' b' a_0' b' a_1' b' a_1' b' a_1' b' a_1' b'.$$

The particular forms of $\varphi(x)$ and x^n implies that $\varphi(x)^{-1}$ can never occur as a subword in the reduced word x^n for all $n \in \mathbb{N}$. Thus, we have that

$$\psi_x^{Br}(x^n) = \sum_{\varphi \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)} f_{\varphi(x)}^{Br}(x^n) \geq f_x^{Br}(x^n) = n.$$

Consequently $\bar{\psi}_x^{Br}(x) \geq 1$ and therefore the homogeneous Aut-invariant quasimorphism $\bar{\psi}_x^{Br}$ is unbounded. □

We note that the above choice of x satisfies $A\text{-code}(x) = (1, 2, 3, 4)$. In fact, any choice for x of the above kind representing a generic code of even length will yield an unbounded Aut-invariant quasimorphism. The following example shoes that some assumption on x is necessary in general.

Example 12.3. Let $A = \mathbb{Z}/3$ and $B = \mathbb{Z}/2$ so that $A*B = \mathrm{PSL}(2, \mathbb{Z})$. Then ψ_w^{Br} is bounded for the cyclically reduced word $w = aba^{-1}b$ where $a \in A$ and $b \in B$ are non-trivial. Up to a cyclic permutation every element g of infinite order in $A * B$ can be written as $g = a_0 b a_1 b \dots a_k b$ for some $k \geq 1$. Then every occurrence of $aba^{-1}b$ in $ba_1 \dots a_k b$ corresponds uniquely to an occurrence of $w^{-1} = baba^{-1}$ in $ba_1 \dots a_k b$. Thus, $|f_w^{Br}(g)| \leq 1$ and therefore $\bar{f}_w^{Br} = 0$. Note that this shows that $\bar{f}_{\theta(w)}^{Br} = 0$ for $\theta \in \mathrm{Aut}(\mathbb{Z}/3)$ as well. Consequently,

$$\bar{\psi}_w^{Br} = \sum_{\varphi \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)} \bar{f}_{\varphi(w)}^{Br} = \bar{f}_w^{Br} + \bar{f}_{\varphi(w)}^{Br} = 0.$$

Appendix B: B_3, $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{PSL}(2, \mathbb{Z})$

The purpose of this appendix is to prove in Proposition 12.6 that the spaces of homogeneous Aut-invariant quasimorphisms on the braid group B_3 and the projective linear group $\mathrm{PSL}(2, \mathbb{Z})$ are in fact isomorphic. Then we generalise this slightly in Proposition 12.7 to include the case of $\mathrm{SL}(2, \mathbb{Z})$. For this we first need the following general lemma about how homogeneous quasimorphisms can descend to quasimorphisms on the quotient of a group by which we mean that they form a commutative diagram through the quotient projection.

Lemma 12.4. *Let $H \leq G$ be a subgroup and $\varphi \colon G \to \mathbb{R}$ be a quasimorphism. If φ is bounded on H, then its homogenisation $\bar{\varphi}$ descends to a homogeneous quasimorphism $\bar{\varphi}_0$ on G/H. Moreover, if φ was Aut(G)-invariant, then $\bar{\varphi}_0$ is invariant under all automorphisms on G/H induced by automorphisms of G.*

Proof. Let $C \geq 0$ be such that the absolute value of φ on H is bounded by C. Let $p \colon G \to G/H$ be the quotient projection. Define $\varphi_0 \colon G/H \to \mathbb{R}$ for $g \in G/H$ by setting $\varphi_0(g) := \varphi(x_g)$ for a chosen preimage $x_g \in p^{-1}(g)$. We claim that φ_0 is a quasimorphism. Let $g, h \in G/H$. Then $p(x_{gh}(x_g x_h)^{-1}) = gh(gh)^{-1} = 1$ and so $x_{gh}(x_g x_h)^{-1} \in H$. We calculate

$$\begin{aligned}
|\varphi_0(gh) - \varphi_0(g) - \varphi_0(h)| &= |\varphi(x_{gh}) - \varphi(x_g) - \varphi(x_h)| \\
&= |\varphi(x_{gh}(x_g x_h)^{-1} x_g x_h) - \varphi(x_g) - \varphi(x_h)| \\
&\leq |\varphi(x_{gh}(x_g x_h)^{-1})| + |\varphi(x_g x_h) - \varphi(x_g) - \varphi(x_h)| + D_\varphi \\
&\leq C + 2D_\varphi.
\end{aligned}$$

Thus φ_0 is a quasimorphism on G/H. Moreover, since φ is bounded on H the homogenisation $\bar{\varphi_0}$ is independent of the particular choice of representative x_g for $g \in G/H$. We claim that $\bar{\varphi_0} \circ p = \bar{\varphi}$. Let $x \in G$. Then

$$\begin{aligned}
|\bar{\varphi}(x) - \bar{\varphi_0}(p(x))| &= \left| \lim_{k\in\mathbb{N}} \frac{\varphi(x^k)}{k} - \lim_{k\in\mathbb{N}} \frac{\varphi_0(p(x)^k)}{k} \right| \\
&= \lim_{k\in\mathbb{N}} \frac{1}{k} \cdot |\varphi(x^k) - \varphi_0(p(x^k))| \\
&= \lim_{k\in\mathbb{N}} \frac{1}{k} \cdot |\varphi(x^k) - \varphi(x_{p(x^k)}(x^k)^{-1}x^k)| \\
&\leq \lim_{k\in\mathbb{N}} \frac{1}{k} \cdot \left(|\varphi(x_{p(x^k)}(x^k)^{-1})| + D_\varphi \right) \\
&= 0
\end{aligned}$$

since $x_{p(x^k)}(x^k)^{-1} \in H$.

Let $f \in \mathrm{Aut}(G)$ be an automorphism that descends to an automorphism $\bar{f}$ of G/H via p and suppose that $\bar{\varphi}$ is invariant under f. We claim that $\bar{\varphi_0}$ is invariant under $\bar{f}$. Let $y \in G/H$ and let $x \in G$ be such that $p(x) = y$. Then

$$\bar{\varphi_0}(\bar{f}(y)) = \bar{\varphi_0}(\bar{f}(p(x))) = \bar{\varphi_0}(p(f(x))) = \bar{\varphi}(f(x)) = \bar{\varphi}(x) = \bar{\varphi_0}(p(x)) = \bar{\varphi_0}(y).$$

□

Let us recall how B_3 forms a central extension of $\mathrm{PSL}(2,\mathbb{Z})$. The standard presentation of the braid group is given by

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

Setting $x = \sigma_1\sigma_2$ and setting $y = \sigma_1\sigma_2\sigma_1$ we see that the braid relation implies $x^3 = y^2$. It holds that $\sigma_1 = x^{-1}y$ and $\sigma_2 = y^{-1}x^2$. Since the relation $x^3 = y^2$ implies the braid relation B_3, we conclude that B_3 admits the alternative presentation

$$B_3 = \langle x, y \mid x^3 = y^2 \rangle.$$

We define $\rho\colon B_3 \to \mathrm{PSL}(2,\mathbb{Z}) = \langle x, y \mid x^3 = 1 = y^2 \rangle$ on generators by $\rho(\sigma_1) = x^{-1}y$ and $\rho(\sigma_2) = y^{-1}x^2$. According to the second presentation of B_3 we have $\ker(\rho) = \langle (\sigma_1\sigma_2)^3 \rangle$ which is the center of B_3 by [Gar69].

Lemma 12.5. *The projection map* $p\colon B_3 \to \mathrm{PSL}(2,\mathbb{Z})$ *which factors out the center of* B_3 *induces a surjective automorphism* $\mathrm{Aut}(B_3) \to \mathrm{Aut}(\mathrm{PSL}(2,\mathbb{Z}))$.

Proof. Since p is surjective, it induces a surjective map on the subgroup of inner automorphisms $\mathrm{Inn}(B_3) \to \mathrm{Inn}(\mathrm{PSL}(2,\mathbb{Z})$. We know that $\mathrm{Out}(\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{Out}(\mathbb{Z}/2 * \mathbb{Z}/3)$ is generated by the non-trivial factor automorphism of $\mathbb{Z}/3$. Indeed, since $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are abelian, all partial conjugations are actually conjugations. Moreover, there are no swap automorphisms and no transvections on $\mathrm{PSL}(2,\mathbb{Z})$.

The non-trivial automorphism ϕ of B_3 is given by inverting the standard generators σ_1 and σ_2. Then with respect to the second presentation we have $\phi(y) = y^{-1}$ and

$$\phi(x) = \sigma_1^{-1}\sigma_2^{-1} = (x^{-1}y)^{-1}(y^{-1}x^2)^{-1} = y^{-1}x^{-1}y.$$

Thus, the automorphism $\bar{\phi} \in \mathrm{Aut}(\mathrm{PSL}(2,\mathbb{Z})$ induced by ϕ is defined by its images on the generators $\bar{\phi}(y) = y$ and $\bar{\phi}(x) = y^{-1}x^{-1}y$. Therefore, $\bar{\phi}$ is equal to the non-trivial factor automorphisms mapping $x \to x^{-1}$ followed by conjugation by y^{-1} on $\mathrm{PSL}(2,\mathbb{Z})$. Consequently, the induced map $p_*\colon \mathrm{Out}(B_3) \to \mathrm{Out}(\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2$ maps the automorphism $\psi \in \mathrm{Out}(B_3)$ to the non-trivial element of $\mathbb{Z}/2$ implying that p is surjective. □

Proposition 12.6. *The spaces of homogeneous Aut-invariant quasimorphisms on* $\mathrm{PSL}(2,\mathbb{Z})$ *and* B_3 *are isomorphic.*

Proof. Every Aut-invariant quasimorphism on $\mathrm{PSL}(2,\mathbb{Z})$ gives rise to an Aut-invariant quasimorphism on B_3 by composition with the projection map according to Lemma 2.23. Conversely, since the non-trivial outer automorphism of B_3 inverts its center, every homogeneous Aut-invariant quasimorphism ψ on B_3 vanishes on $Z(B_3) = \langle(\sigma_1\sigma_2)^3\rangle$. By Lemma 12.4 ψ descends to a homogeneous quasimorphism $\bar{\psi}$ invariant under all automorphisms on $\mathrm{PSL}(2,\mathbb{Z})$ that are induced by automorphisms of B_3. By Lemma 12.5 this means that $\bar{\psi}$ is in fact invariant under all automorphisms of $\mathrm{PSL}(2,\mathbb{Z})$. The construction given in Lemma 12.4 is an inverse to the precomposition with the projection map. □

Consider the amalgamated product $G_q = \mathbb{Z}/2q *_{\mathbb{Z}/q} \mathbb{Z}/3q$ for $q \geq 2$ where $Z/q \leq \mathbb{Z}/pq$ via multiplication by p for $p = 2, 3$. Recall that $\mathrm{SL}(2,\mathbb{Z}) = G_2$. Then $Z(G_q) = \mathbb{Z}/q$ is the center of G_q, since the quotient $G_q/(\mathbb{Z}/q) = \mathbb{Z}/2 * \mathbb{Z}/3 = \mathrm{PSL}(2,\mathbb{Z})$ has trivial center. Moreover. the automorphism $\iota \in \mathrm{Aut}(G_p)$ defined by inverting the generators of both cyclic factors inverts the center and descends to the non-trivial factor automorphism on $\mathbb{Z}/2 * \mathbb{Z}/3$. Thus, the map $\mathrm{Aut}(G_q) \to \mathrm{Aut}(\mathrm{PSL}(2,\mathbb{Z}))$ induced by the projection map is surjective. Repeating the argument of the proof of Proposition 12.6 before, we conclude the following proposition.

Proposition 12.7. *The spaces of homogeneous Aut-invariant quasimorphisms on* $\mathrm{PSL}(2,\mathbb{Z})$*,* B_3 *and* G_q *are isomorphic for all* $q \geq 1$. □

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