Braids in dynamics

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Abstract

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1 Introduction

The present series of lectures is concerned with a function on a connected group of volume preserving diffeomorphisms of a manifold M defined by the following formula:

$$\boxed{f \mapsto \int_{\mathrm{Conf}_n(M)} \psi(\gamma(f,x))dx \qquad (1.1)}$$

where

- $f \in \mathrm{Diff}_0(M, \mathrm{vol})$ is a volume preserving diffeomorphism isotopic with the identity.
- $\mathrm{Conf}_n(M)$ is the space of ordered configurations of n points in M; if M is a surface then $n \in \mathbf{N}$ is an arbitrary positive integer, while if $\dim(M) \geq 3$ then $n = 1$.
- $\psi \colon \pi_1(\mathrm{Conf}_n(M)) \to \mathbf{R}$ is a ***quasi-morphism*** on the fundamental group of the configuration space; if M is a surface then this fundamental group is called the ***braid group*** of M on n strands; otherwise, it is just the fundamental group of M.
- $\gamma \colon \mathrm{Diff}_0(M) \times \mathrm{Conf}_n(M) \to \pi_1(\mathrm{Conf}_n(M))$ is a certain, geometrically defined, ***cocycle***.

The function defined above serves as a motivation to learn about braid groups, groups of diffeomorphisms, quasi-morphisms and a few other things. This construction first appeared in a paper by Gambaudo and Ghys [13] and was subsequently developed and applied in many contexts by Brandenbursky and his collaborators [2, 4, 3, 8, 9].

What is the question? Given a manifold M we want to understand its group of diffeomorphisms $\mathrm{Diff}(M)$. This is done on many levels: group theoretic, topological, geometric. One way of doing it is to consider diffeomorphisms up to isotopy. That is, two diffeomorphisms f_0 and f_1 are equivalent if there is a continuous path $\{f_t\}$ between them. In other words, we study the *group* of connected components of $\mathrm{Diff}(M)$, which is called the mapping class group. In this approach the connected component of the identity $\mathrm{Diff}_0(M)$ is reduced to just one, trivial, element. In this course, however, we are interested in the connected group $\mathrm{Diff}_0(M)$ and its subgroup of volume preserving diffeomorphisms.

So what do we do? Let $f \in \mathrm{Diff}_0(M)$. We can *choose* an isotopy $\{f_t\}$ from the identity $f_0 = \mathrm{Id}$ to $f = f_1$. Then we pick a point $x \in M$ and evaluate this isotopy to obtain a path $t \mapsto f_t(x)$. By analysing such paths we want to understand something about f and the whole group $\mathrm{Diff}_0(M)$. There are immediate problems: what is the dependence on the choice of an isotopy? What is the dependence on the choice of $x \in M$? If $\{f'_t\}$ is another isotopy from Id to f and if the paths $\{f_t\}$ and $\{f'_t\}$ are homotopic relative to the endpoints then the paths $\{f_t(x)\}$ and $\{f'_t(x)\}$ are also homotopic relative to the endpoints which is good! But what if the isotopies are not homotopic?

Why choose just one point $x \in M$? Why not several of them? By evaluating the isotopy $\{f_t\}$ on a bunch of points $x_1, \ldots, x_n$ and thinking of the parameter $t \in [0,1]$ as time, we can see the evaluations $t \to f_t(x_i)$ as movements of points. This is how braids arise from this approach. The complexity of those movements, or braids, can tell us something about f. This is what this course is about and we are going to prove that formula (1.1) defines an unbounded quasi-morphism on the group of volume preserving diffeomorphisms of a manifold M.

Prerequisites. It is essential for this course to be fluent with the standard material of undergraduate mathematics and the *fundamental group* plus a little bit more of group theory: generators and relations, presentations. Some parts of the course require basic differential geometry: manifolds, vector fields, differential forms, Stokes's Lemma, Cartan's magic formula:

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega,$$

and a little bit of homology and cohomology theory.

What to do if you are missing a prerequisite? Take it easy. In most of the cases, I can explain a difficult bit in simpler terms. Ask for an explanation. For example, the flux homomorphism $\mathrm{Flux}\colon \widetilde{\mathrm{Diff}}_0(M,\mathrm{vol}) \to H^{n-1}(M;\mathbf{R})$ has values in the $(n-1)$-th cohomology of M. If you don't know what cohomology is, think of flux as a *certain* homomorphism with values in a finite dimensional vector space; and remember the theorem that its kernel is a simple group. Moreover, I will define flux by drawing pictures and knowing what cohomology is irrelevant.

Exercises. These notes are mostly exercises. Some of them are trivial, some of them are harder, and there are a few whose aim is to fill gaps in literature. There are too many of them and I don't think we can solve all of them. I hope we can solve the most interesting ones and include solutions in these notes.

Contents

2 Braid groups

2.1 Configuration spaces

Let X be a topological space and let

$$\begin{aligned} \mathrm{C}_n(X) &= \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j, \text{ for } i \neq j\} \\ \mathrm{S}_n(X) &= \{\{x_1, \dots, x_n\} \subseteq X^n \mid \#\{x_1, \dots, x_n\} = n\} \end{aligned}$$

denote the configuration spaces. The first one is the space of n-tuples of points in X, while the second is the space of subsets of X of cardinality n. The topology of $\mathrm{C}_n(X)$ is induced from X^n and the topology of $\mathrm{S}_n(X)$ is the quotient topology (see the first exercise below).

Exercise 2.2. The purpose of this exercise is to show that configuration spaces are complicated.

1. Observe that $\mathrm{S}_n(X) = \mathrm{C}_n(X)/\mathbf{S}_n$, where $\mathbf{S}_n$ denotes the symmetric group on n letters. Give conditions under which the quotient map $\mathrm{C}_n(X) \to \mathrm{S}_n(X)$ is a covering.
2. Let $X = [0, 1]$. Draw the configuration spaces for $n = 2, 3$.
3. Let Y be the tripod (a graph which looks like the letter Y). Draw the configuration spaces for $n = 2, 3$. Try to do it for the graph which looks like the letter X. What is the homotopy type of these spaces?
4. The same for the circle $\mathbf{S}^1$.

2.3 Braids and braid groups

Definition 2.4. Let X be a path-connected space. The fundamental group $\pi_1(\mathrm{C}_n(X))$ is called the ***pure braid group*** of X on n strings and it is denoted by $\mathbf{P}_n(X)$. The fundamental group $\pi_1(\mathrm{S}_n(X))$ is called the ***braid group*** of X on n strings and it is denoted by $\mathbf{B}_n(X)$. An element of $\mathbf{B}_n(X)$ is called a braid on X, while an element of $\mathbf{P}_n(X)$ is called a pure braid. If $X = \mathbf{D}^2$ is the two-dimensional disc, the braid groups are denoted by $\mathbf{B}_n$ and $\mathbf{P}_n$, respectively.

Exercise 2.5.

1. Let Σ be a connected surface. Create a number of different mental pictures of a (pure) braid on Σ. For example, it can be a disjoint union of suitable curves or their images in $\Sigma \times [0, 1]$, or a movement of particles on Σ without collisions. Draw a complicated braid representing the trivial element in the braid group.
2. Draw a non-trivial braid in the centre $Z(\mathbf{B}_n)$ of the braid group.
3. Let $\sigma_i \in \mathbf{B}_n$ be the braid swapping the i-th and the $(i+1)$-th strings, where $1 \leq i \leq n-1$. Show (by drawing) that σ_i and σ_j are conjugate.
4. Show that there is a surjective homomorphism $\mathbf{P}_n(X) \to \mathbf{P}_m(X)$, for $m \leq n$.

5. Let $p\colon \mathrm{C}_n(\Sigma) \to \mathrm{C}_{n-1}(\Sigma)$ be the map defined by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$. What is the preimage of a point? Is p a fibre bundle?

6. Let $\mathbf{D}^2 \to \Sigma$ be an injective map. Observe that it defines a homomoprhism $\mathbf{B}_n \to \mathbf{B}_n(\Sigma)$. Is it injective? When it is? The same for pure braid groups. Draw braids in the kernel.

7. Prove that there is a diffeomorphism
$$\mathrm{C}_2(\mathbf{T}^2) \cong \mathbf{T}^2 \setminus \{(0,0)\} \times \mathbf{T}^2.$$
Deduce that $\mathbf{P}_2(\mathbf{T}^2) \cong \mathbf{F}_2 \times \mathbf{Z}^2$. Draw generators of $\mathbf{P}_2(\mathbf{T}^2)$.

8. Is 'configuration space' a functor? Is 'braid group' a functor? On which categories it is?

Exercise 2.6. Let M be a manifold of dimension at least 3. Show that $\mathbf{P}_n(M) \cong \pi_1(M)^n$. For this reason we don't discuss braid groups on manifolds of higher dimension. The fundamental group contains all the information.

Definition 2.7. Let Σ be a connected surface and let $z = (z_1, \ldots, z_n) \in \mathrm{C}_n(\Sigma)$. A ***geometric braid*** on Σ based at z is the image of a smooth injective map $\beta\colon [0,1] \times \{1, \ldots, n\} \to [0,1] \times \Sigma$ such that the composition of β followed by the projection onto $[0,1]$ the interval is the identity and
$$\beta(0, \{1, \ldots, n\}) = \beta(1, \{1, \ldots, n\}) = \{z_1, \ldots, z_n\}.$$

Exercise 2.8.

1. Define a pure geometric braid.

2. Define an equivalence relation on the set of geometric braids such that its equivalence classes are in a bijective correspondence with elements of the braid group.

Exercise 2.9. This exercise shows examples where braid groups appear in 'nature'.

1. Let $\mathrm{Poly}_n(\mathbf{C})$ denote the space of complex polynomials of degree at most n. Let $\Delta_n(\mathbf{C}) \subseteq \mathrm{Poly}_n(\mathbf{C})$ be the subset consisting of polynomials with multiple roots. Let $\mathrm{Poly}_n(\mathbf{C}) \setminus \Delta_n(\mathbf{C}) \to \mathrm{S}_n(\mathbf{C})$ be the map sending a polynomial to the set of its roots. Show that this map is a homeomorphism. Contemplate consequences of this fact.

2. Let $H_{i,j} = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid z_i = z_j\}$. It is a hyperplane in $\mathbf{C}^n$. Let $H = \bigcup_{i \neq j} H_{i,j}$ be the union of these hyperplanes. Observe that $\mathbf{C}^n \setminus H \cong \mathrm{C}_n(\mathbf{C}) \cong \mathrm{C}_n(\mathbf{D}^2)$. In particular, the pure braid group is the fundamental group of the complement of the hyperplane arrangement H.

3. Let M be a manifold and let $z = (z_1, \ldots, z_n) \in \mathrm{C}_n(M)$. Let $f\colon M \to M$ be a diffeomorphism of M which is isotopic to the identity and such that $f(z_i) = z_i$ for $i = 1, \ldots, n$. Let $\{f_t\}$ an isotopy from the identity to f. Observe that the evaluation $f_t(z_1, \ldots, z_n)$ defines a pure braid on M. Give an example of M and f such that this braid depends on the choice of an isotopy.

2.10 Algebraic properties of standard braid groups

Theorem 2.11 (Artin)**.** *The braid group* $\mathbf{B}_n$ *has the following presentation*

$$\mathbf{B}_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i, & \textit{if } |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & \textit{for } i = 1, \ldots, n-2. \end{array} \right\rangle.$$

Exercise 2.12.

1. Show that $\mathbf{B}_2 \cong \mathbf{P}_2 \cong \mathbf{Z}$ and that $\mathbf{P}_n \subseteq \mathbf{B}_n$ is a subgroup of index 2.

2. Show that there is an extension
$$1 \to \mathbf{P}_n \to \mathbf{B}_n \to \mathbf{S}_n \to 1.$$

3. A normal subgroup H of a group G is said to be ***normally generated*** by a set $S \subseteq G$ if H is the smallest normal subgroup of G containing S. Show that $\mathbf{B}_n$ is normally generated by $\{\sigma_1\}$.

4. Show that $\sigma_i \to 1$ for $i = 1, \ldots, n-1$ defines a surjective homomorphism $\mathbf{B}_n \to \mathbf{Z}$. Show that this is the abelianisation of $\mathbf{B}_n$, for $n \geq 2$.

5. Removing the last string defines a surjective homomorphism $\mathbf{P}_n \to \mathbf{P}_{n-1}$. Show that it has a section and determine its kernel.

6. Write a presentation of the *full twist* braid in terms of generators σ_i. Verify that it is indeed central. Show that it generates the centre of both $\mathbf{P}_n$ and $\mathbf{B}_n$; see [14, Theorem 1.24] for the proof.

7. Show that
$$\sigma_1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \qquad \sigma_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$$
defines a surjective homomorphism $\mathbf{B}_3 \to \mathrm{SL}(2, \mathbf{Z})$. Determine its kernel.

8. Show that $\mathbf{P}_3 \cong \mathbf{F}_2 \times \mathbf{Z}$. Here $\mathbf{F}_2$ denotes the free group of rank 2.

9. Show that $\mathbf{B}_3 = \langle a, b \mid a^2 = b^3 \rangle$. That's the fundamental group of the complement of the trefoil knot!

10. Show that $\mathbf{B}_n$ is generated by two elements σ_1 and $\alpha = \sigma_1 \ldots \sigma_{n-1}$. Hint: conjugate σ_1 by a power of α and see what happens.

Exercise 2.13. The aim of this exercise is to show that the commutator subgroup $[\mathbf{B}_n, \mathbf{B}_n]$ of the braid group is perfect for $n \geq 5$. We start with a general observation, due to Orevkov [16, Lemma 1.1].

Lemma 2.14. *Let G be a group and let $S \subseteq G$ be a subset. Let Γ_2 be a graph with vertices S and an edge between s and t if an only if $[s, t] = 1$. Let $\pi \colon G \to \mathbf{Z}$ be defined by $\pi(s) = 1$ for every $s \in S$. Let $K = \ker \pi$. Show that if the graph Γ_2 is connected then $[K, K] = [G, G]$.*

1. Show that $[K,K]$ is normal in G.

2. Show that the lemma is trivially true for $G = \mathbf{Z}$. In what follows assume that S has at least two elements.

3. The inclusion $[K,K] \subseteq [G,G]$ is obvious. Show that in order to prove the opposite inclusion it is enough to show that $[s,t] \in [K,K]$ for every $s,t \in S$.

4. Verify the following identity:

$$[x,z] = [x,y]\left[yxy^{-2}, yzy^{-2}\right][y,z].$$

5. For $s,t \in S$ let $s \simeq t$ if and only if $[s,t] \in [K,K]$. Show that it is an equivalence relation.

6. Finish the proof of the lemma.

7. Deduce that $[\mathbf{B}_n, \mathbf{B}_n]$ is perfect for $n \geq 5$ including $n = \infty$.

8. Learn what is an Artin group in general and state and prove a theorem on the perfectness of its commutator subgroup.

3 The Gambaudo-Ghys construction

3.1 The Gambaudo-Ghys cocycle

Let M be a manifold, $z = (z_1, \ldots, z_n)$ a tuple of base-points, and let $f\colon M \to M$ be a diffeomorphism isotopic to the identity. We saw in Exercise 2.9.3 that if f preserves the base-points then an isotopy defines a braid on M. In what follows, we present a trick allowing to relax this assumption.

Given a tuple $x = (x_1, \ldots, x_n) \in \mathrm{C}_n(M)$ choose a set of paths $\ell_{x_i}\colon [0,1] \to M$ such that $\ell_{x_i}(0) = z_i$ and $\ell_{x_i}(1) = x_i$. For example, equip M with an auxiliary Riemannian metric and choose ℓ_{x_i} to be geodesics of minimal length. If M is non-positively curved then the choice is unique. Otherwise, we make a choice. For example, if $M = \mathbf{D}^2$ then ℓ_{x_i} are straight intervals.

Let $\gamma\colon \mathrm{Diff}_0(M) \times \mathrm{C}_n(M) \to \mathbf{P}_n(M)$ be defined by

$$\gamma(f,x) = \left[\ell_x * \{f_t(x)\} * \overline{\ell_{f(x)}}\right].$$

That is, $\gamma(f,x)$ is a braid obtained by concatenation of the following paths: go from the base-point z to x via ℓ_x, then evaluate the isotopy $\{f_t\}$ on x, then go from $f(x)$ to the base-point. This is, of course, all wrong.

Problem 1: Geodesics from $z = (z_1, \ldots, z_n)$ to $x = (x_1, \ldots, x_n)$ might intersect and then $\gamma(f,x)$ is not a braid. However, for a generic x they do not intersect. By this we mean that there exists a set of full measure in $\mathrm{C}_n(M)$ consisting of those x for which $\gamma(f,x)$ is indeed a braid. Recall the formula from the first page: it is defined by an integral so having a bad set of measure zero is fine.

Problem 2: There is a choice of an isotopy and it is not clear at all what is the dependence of $\gamma(f,x)$ on that choice. Suppose we have two isotopies $\{f_t\}$ and $\{f'_t\}$ from the identity to f. If they are homotopic relative to the endpoints then they define the same braid $\gamma(f,x)$. This happens if the group $\mathrm{Diff}_0(M)$ is simply connected. This holds, for example, if M is a surface of genus at least 2 or if $M = \mathbf{D}^2$. The following exercise clarifies the situation in greater generality.

Exercise 3.2. Let G be a path-connected topological group acting continuously on a path-connected space X. Let $x \in X$ be a base-point and let $\mathrm{ev}_x \colon G \to X$ be the ***evaluation map*** defined by

$$\mathrm{ev}_x(f) = f(x).$$

Consider the induced homomorphism $(\mathrm{ev}_x)_* \colon \pi_1(G, \mathrm{Id}) \to \pi_1(X,x)$. Show that the image of $(\mathrm{ev}_x)_*$ is contained in the centre of $\pi_1(X,x)$. *Hint: Observe that two elements* $[\gamma_1], [\gamma_2] \in \pi_1(X,x)$ *commute if and only if the map* $\gamma_1 \vee \gamma_2 \colon \mathbf{S}^1 \vee \mathbf{S}^1 \to X$ *extends to the torus* $\mathbf{S}^1 \times \mathbf{S}^1$.

In our situation we consider the action of $\mathrm{Diff}_0(M)$ on the configuration space $\mathrm{C}_n(M)$. In full generality, we can only claim that the braid $\gamma(f,x)$ is defined modulo the centre of $\mathbf{P}_n(M)$. This is still good because in most cases the quotient group $\mathbf{P}_n(M)/Z(\mathbf{P}_n(M))$ is still big enough to define meaningful invariants. Also, very often we will know that the centre of $\mathbf{P}_n(M)$ is trivial.

Proposition 3.3. *The function* $\gamma \colon \mathrm{Diff}_0(M) \times \mathrm{C}_n(M) \to \mathbf{P}_n(M)$ *satisfies the following* ***cocycle*** *identity*

$$\gamma(fg,x) = \gamma(g,x)\gamma(f,g(x)).$$

Exercise 3.4. Prove the above proposition.

3.5 The construction with a homomorphism

Let $\psi \colon \mathbf{P}_n(M) \to \mathbf{R}$ be a homomorphism. Choose a volume form on M and consider the integral

$$\int_{\mathrm{C}_n(M)} \psi(\gamma(f,x))dx,$$

with respect to the product measure on $\mathrm{C}_n(M) \subseteq M^n$. First of all, it is not clear whether the function $x \mapsto \psi(\gamma(f,x))$ is integrable. We will deal with it later. Let's investigate how this integral behaves on the composition of two diffeomorphisms.

$$\begin{aligned}
&\int_{\mathrm{C}_n(M)} \psi(\gamma(fg,x))dx \\
&\qquad = \int_{\mathrm{C}_n(M)} \psi(\gamma(g,x)\gamma(f,g(x))dx && \texttt{by the cocycle property} \\
&\qquad = \int_{\mathrm{C}_n(M)} \big(\psi(\gamma(g,x)) + \psi(\gamma(f,g(x))\big)dx && \psi\ \texttt{is a homomorphism} \\
&\qquad = \int_{\mathrm{C}_n(M)} \psi(\gamma(g,x))dx + \int_{\mathrm{C}_n(M)} \psi(\gamma(f,g(x))dx \\
&\qquad = \int_{\mathrm{C}_n(M)} \psi(\gamma(g,x))dx + \int_{\mathrm{C}_n(M)} \psi(\gamma(f,x))dx && \texttt{change of variables}
\end{aligned}$$

The last step is problematic and can be done only if g preserves the volume. This can't be avoided. Moreover, we want the integral to be finite. So we restrict to diffeomorphisms with compact supports. We have thus proved the following result.

Proposition 3.6. *If $\psi\colon \mathbf{P}_n(M)$ is a homomorphism then the formula*

$$f \mapsto \int_{\mathrm{C}_n(M)} \psi(\gamma(f,x))dx$$

defines a homomorphism on the group $\mathrm{Diff}_0^c(M,\mathrm{vol})$ of compactly supported volume preserving diffeomorphisms isotopic with the identity.

Problem: Groups $\mathrm{Diff}_0^c(M,\mathrm{vol})$ are often simple or very close to being simple, in the sense that the space of homomorphisms $\mathrm{Hom}(\mathrm{Diff}_0^c(M,\mathrm{vol}),\mathbf{R})$ is finite dimensional, and it is generated by well known homomorphisms. If $\mathrm{Diff}_0^c(M,\mathrm{vol})$ admits non-trivial homomorphisms, their kernels form still a very big subgroup which is simple. These are the groups we want to understand.

3.7 The volume flux

Exercise 3.8. Let (M,vol) be an n-dimensional manifold equipped with a volume form. Let $S\subseteq M$ be a closed oriented submanifold of dimension $n-1$. Let $\{f_t\}$ be an isotopy from the identity to $f\in \mathrm{Diff}_0^c(M,\mathrm{vol})$ and let $F\colon S\times[0,1]\to M$ be defined by

$$F(s,t)=f_t(s).$$

Let $\widetilde{\mathrm{Flux}}\colon \widetilde{\mathrm{Diff}}{}_0^c(M,\mathrm{vol})\to H^{n-1}(M;\mathbf{R})=\mathrm{Hom}(H_{n-1}(M;\mathbf{R}),\mathbf{R})$ be a map defined by

$$\langle \widetilde{\mathrm{Flux}}(f_t),[S]\rangle = \int_{S\times[0,1]} F^*\mathrm{vol}.$$

1. Show that $\widetilde{\mathrm{Flux}}$ is a homomorphism. In this exercise we use the fact that a real homology class can be represented by a cycle which is a submanifold.
2. Show that $\widetilde{\mathrm{Flux}}$ induces a homomorphism

   $$\mathrm{Flux}\colon \mathrm{Diff}_0^c(M,\mathrm{vol})\to H^{n-1}(M;\mathbf{R})/\Gamma,$$

   where Γ is the image with respect to $\widetilde{\mathrm{Flux}}$ of the fundamental group of $\mathrm{Diff}_0^c(M,\mathrm{vol})$ viewed as a subgroup of the universal cover $\widetilde{\mathrm{Diff}}{}_0^c(M,\mathrm{vol})$.
3. Let $\{f_t\}\subseteq \mathrm{Diff}_0^c(M,\mathrm{vol})$ be an isotopy from the identity to f. Let X_t be a family of vector fields defined by

   $$X_t(f_t(x))=\frac{d}{ds}\Big|_{s=t} f_s(x)$$

   (a) Show that $L_{X_t}\mathrm{vol}=0$. Deduce that $\iota_{X_t}\mathrm{vol}$ is a closed $(n-1)$-form.

   (b) Show that $\widetilde{\mathrm{Flux}}(\{f_t\})=\int_0^1[\iota_{X_t}\mathrm{vol}]dt$.

Exercise 3.9. In an unpublished paper, Thurston sketched a proof that if M is a closed manifold of dimension at least 3 then the kernel $\ker \mathrm{Flux} \subseteq \mathrm{Diff}_0^c(M, \mathrm{vol})$ is a simple group. The full argument is implicitly given in Banyaga's book. Write an argument proving Thurston's theorem as a sequence of results with precise references to the relevant statements in Banyaga's book.

The kernel of Flux is also simple if M is a closed surface, which is a theorem of Banyaga. It is also know that groups of all diffeomorphisms (not necessarily volume preserving) isotopic with the identity are simple, which is due to Herman, Mather and Thurston.

Notice that if G is a simple group and $1 \neq f \in G$ is a non-trivial element then every other element $g \in G$ is a product of conjugates of f and its inverse f^{-1}. In other words the union of conjugacy classes $\mathrm{Conj}(f) \cup \mathrm{Conj}(f^{-1})$ generates G. In particular, if G is a simple group of diffeomorphisms of M, you can pick your favourite diffeomorphism f and every other diffeomorphism is a composition of conjugates of $f^{\pm 1}$. You may ask how many? Is there a uniform bound? You might have a whole family of favourite diffeomorphisms. For example, those generated by vector fields. You could ask similar questions.

3.10 Simplicity of $\mathrm{Diff}_0(M)$

The group $\mathrm{Diff}_0(M)$ of compactly supported diffeomorphisms is simple. This is also due to Thurston (mostly) and the perfectness is the difficult step. Recently, Haller, Rybicki and Teichmann gave a simpler proof of perfectness, which has been digested to a 6 page paper by Mann in [15].

Exercise 3.11.

1. Show that $\mathrm{Diff}_0(M)$ has the *fragmentation property*. More precisely, show that for every $f \in \mathrm{Diff}_0(M)$ there exist $n \in \mathbf{N}$ and $g_1, \ldots, g_n \in \mathrm{Diff}_0(M)$ such that $\mathrm{supp}(g_i)$ is diffeomorphic to a ball and $f = g_1 \ldots g_n$. See [**?**, Lemma 2.1.8] for a solution.

2. Let G be a group. Suppose that for any non-trivial $g \in G$ there exists a subgroup $H \subseteq G$ such that

   (a) H and $g^{-1}Hg$ commute;

   (b) $[H, H]$ normally generates G.

   Show that g normally generates G and hence G is simple. Hint: Show first that if $a, b \in H$ then $[[a, g], b] = [a, b]$.

3. Let $B \subseteq M$ be a ball and let $H = \mathrm{Diff}_0(B) \subseteq \mathrm{Diff}_0(M) = G$ be the subgroup consisting of diffeomorphisms supported in B. Show that the previous item applies, due to the fragmentation property and the perfectness of H.

3.12 Simplicity of homeomorphisms

Exercise 3.13. Show that the group $\mathrm{Homeo}_0(\mathbf{R}^n)$ is simple.

1. Let $B_i \subseteq \mathbf{R}^n$, $i \in \mathbf{N}$, be a family of pairwise disjoint balls of radius 2^{-i} such that their union is contained in a compact set. Construct such a family and a homeomorphism $f \in \mathrm{Homeo}_0(\mathbf{R}^n)$ such that $f(B_i) = B_{i+1}$.

2. Let $g \in \mathrm{Homeo}_0(\mathbf{R}^n)$ be supported in B_0. Let $h = \prod_{i=0}^{\infty} f^i g f^{-1}$. Show that $fhf^{-1} = g^{-1}h$. Deduce that $\mathrm{Homeo}_0(\mathbf{R}^n)$ is perfect. This exercise is sometimes referred to as *Anderson's trick*.

3. Let $s' \in \mathrm{Homeo}_0(\mathbf{R}^n)$ be a non-trivial homeomorphism. Show that a conjugate s of s' displaces B_0. That is, $s(B_0) \cap B_0 = \emptyset$.

4. Let $g' \in \mathrm{Homeo}_0(\mathbf{R}^n)$ be any homeomorphism. Show that a conjugate g of g' is supported in B_0.

5. Show that $g = [f,h] = [[f,s],h]$ and deduce that $\mathrm{Homeo}_0(\mathbf{R}^n)$ is simple.

As in the case of diffeomorphisms, the simplicity of $\mathrm{Homeo}_0(M)$ follows from the fragmentation property, due to Edwards-Kirby [12], and the perfectness of $\mathrm{Homeo}_0(\mathbf{R}^n)$ shown above.

4 Quasi-morphisms

4.1 Definitions

Let G be a group. A function $\psi\colon G \to \mathbf{R}$ is called a quasi-morphism if there exists a number $D \geq 0$ such that

$$|\psi(h) - \psi(gh) + \psi(g)| \leq D,$$

for all $g, h \in G$. Moreover, ψ is called ***homogeneous*** if $\psi(g^n) = n\psi(g)$ for all $g \in G$ and $n \in \mathbf{Z}$. The smallest number D satisfying the defining inequality is called the ***defect*** of ψ; it is denoted by D_ψ.

Exercise 4.2. Let $\psi\colon G \to \mathbf{R}$ be a quasi-morphism.

1. Let $\psi'(g) = \frac{1}{2}\left(\psi(g) - \psi(g^{-1})\right)$. Show that ψ' is a quasi-morphism and $D_{\psi'} \leq D_\psi$.

2. Recall Fekete's Lemma and show that

   $$\overline{\psi}(g) = \lim_{n\to\infty} \frac{\psi(g^n)}{n}$$

   is a well defined quasi-morphism. It is called the homogenisation of ψ.

3. Show that $|\overline{\psi}(g) - \psi(g)| \leq D_\psi$ and estimate the defect of the homogenisation.

4. Show that the homogenisation is indeed homogeneous.

5. Show that a homogeneous quasi-morphism is constant on conjugacy classes.

6. Two quasi-morphisms are called equivalent if $\sup|\psi(g)-\varphi(g)|<\infty$. In particular, the homogenisation is a nice representative of the equivalence class of ψ. Consider the set $Q(\mathbf{Z},\mathbf{Z})$ all quasi-morphisms $\psi\colon \mathbf{Z}\to\mathbf{Z}$. Equivalence classes admit addition (defined obviously) and multiplication defined by composition. The map

$$[\psi]\mapsto\overline{\psi}(1)$$

defines an isomorphism $Q(\mathbf{Z},\mathbf{Z})/\approx$ with the real numbers. In fact, real numbers can be defined like this!

4.3 Examples

Exercise 4.4.

1. Let $G=\mathrm{PSL}(2,\mathbf{Z})=\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/3\mathbf{Z}=\langle a,b\mid a^2=1=b^3\rangle$. Let $\psi\colon G\to\mathbf{Z}$ be defined by

$$\psi(ab^{\epsilon_1}ab^{\epsilon_2}\dots ab^{\epsilon_n}a)=\sum_{i=1}^{n}\epsilon_i,$$

where $\epsilon_i=\pm1$ and the first and the last a in the argument are optional. Show that ψ is an unbounded quasi-morphism and estimate its defect. Notice that G does not admit non-trivial real-valued homomorphisms.

2. Deduce that the braid group $\mathbf{B}_3$ admits an unbounded quasi-morphism.

3. Show that any homogeneous quasi-morphism on an abelian group is a homomorphism.

4. Let $\mathbf{F}$ be a non-abelian free group and let $w\in F$ be a reduced word. Let $c_w(g)$ be the maximal number of *disjoint* copies of w in a reduced representative of g. Let $\psi_w(g)=c_w(g)-c_{w^{-1}}(g)$. Show that ψ_w is a quasi-morphism and estimate its defect. It is called the small counting quasi-morphism.

5. Similarly, $C_w(g)$ is the maximal number of copies of w in the reduced representative of g and $\Psi_w(g)=C_w(g)-C_{w^{-1}}(g)$. Show that Ψ_w is a quasi-morphism and estimate its defect. It is called the big counting quasi-morphism.

6. Deduce that the space of quasi-morphisms on a non-abelian free group is infinite dimensional. Deduce that the space of quasi-morphisms on a pure braid group $\mathbf{P}_n$ is infinite dimensional for $n\geq 3$.

7. Let M be a closed hyperbolic manifold and let $\alpha\in\Omega^1(M)$ be a 1-form. Let $\psi_\alpha\colon\pi_1(M,x)\to\mathbf{R}$ be defined by

$$\psi_\alpha(g)=\int_\gamma\alpha,$$

where γ is the unique geodesic arc based at x representing g. Show that ψ_α is a quasi-morphism and estimate its defect. It is called the ***de Rham*** quasi-morphism. Express the homogenisation of ψ_α geometrically. Deduce that the space of quasi-morphisms on $\pi_1(M)$ is infinite dimensional.

8. A word $g \in \mathbf{F}_2$ can be represented by a continuous, piecewise horizontal/vertical path on the plane with turning point on the integer grid $\mathbf{Z}^2$. It g is reduced then the path has no back-tracks. Let $\xi \colon \mathbf{F}_2 \to \mathbf{Z}$ be a function defined by

$$\xi(g) = R(g) - L(g),$$

where $R(g)$ is the number of right turns of the representing path and $L(g)$ is the number of left turns. Show that ξ is a quasi-morphism and compute its defect. Express ξ as a combination of counting quasi-morphisms. It is called ***the snake***.

4.5 Why quasi-morphisms are great?

Suppose a group G is generated by a set $S \subseteq G$. Define the ***word norm*** associated with S by

$$|g|_S = \min\{k \in \mathbf{N} \mid g = s_1 \dots s_k,\ s_i \in S \cup S^{-1}\}.$$

The corresponding metric is defined by

$$d_S(g,h) = |gh^{-1}|_S.$$

Exercise 4.6.

1. Prove that the word norm satisfies the following properties:
   - $|g|_S \geq 0$;
   - $|g|_S = 0$ if and only if $g = 1$;
   - $|g^{-1}|_S = |g|_S$;
   - $|gh|_S \leq |g|_S + |h|_S$.
2. Prove that d_S is a right-invariant metric.
3. Let $\psi \colon G \to \mathbf{R}$ be a quasi-morphism which is bounded on S. Show that ψ is Lipschitz with respect to d_S. That is, show that there exists a constant $L > 0$ such that

   $$d_S(g,h) \leq L|\psi(g) - \psi(h)|,$$

   for all $g, h \in G$. Deduce that if ψ is, moreover, unbounded then d_S has infinite diameter. Prove that there exists an element $g \in G$ such that $|g^n|_S \geq Cn$, for some $C > 0$.
4. Let $S = \{[g,h] \in G \mid g,h \in G\}$ be the set of all commutators in G. It generates the commutator subgroup $[G,G] \subseteq G$ and the corresponding word norm is called the ***commutator length***. Show that if G admits an quasi-morphism which is unbounded on $[G,G]$ then the commutator length has infinite diameter.
5. Show that every element of $[\mathbf{B}_\infty, \mathbf{B}_\infty]$ - the commutator subgroup of the braid group in infinitely many strings - is a product of two commutators. What can you say about quasi-morphisms on $\mathbf{B}_\infty$?

6. Let $S \subseteq G$ be a finite subset. Let $\overline{S} = \bigcup_{g\in G} g^{-1}S^{\pm 1}g$ be the set of all conjugates of elements from $S^{\pm 1} = S \cup S^{-1}$. Suppose that $\overline{S}$ generates G. Show that the word norm associated with $\overline{S}$ is invariant under conjugations and that the corresponding word metric is bi-invariant. Show that if G admits an unbounded quasi-morphism then the above bi-invariant metric has infinite diameter.

7. Show that the above exercise applies to simple groups.

8. Show that the orthogonal group SO(3) does not admit unbounded quasi-morphisms.

9. Let $S \subseteq \mathbf{F}_2$ be the set of all palindromes - words that read the same from left to right and from right to left. The corresponding word norm is called the ***palindromic length***. Show that it has infinite diameters.

10. Let $S \subseteq \mathbf{F}_2 = \langle a, b\rangle$ be the set of all elements of the form $s = \alpha(a)$, where $\alpha \in \operatorname{Aut}(\mathbf{F}_2)$ is an automorphism of $\mathbf{F}_2$. Such element is called primitive. The corresponding word norm is called the ***primitive length***. Show that it has infinite diameter.

Definition 4.7. A group G is called ***bounded*** if every conjugation-invariant norm has finite diameter. Otherwise it is called ***unbounded***.

Exercise 4.8.

1. Suppose G is generated by a set S which union of finitely many conjugacy classes. Show that G is bounded if and only if the word metric associated with S has finite diameter.

2. Show that if $\psi\colon G \to \mathbf{R}$ is a non-trivial homogeneous quasi-morphism then G is unbounded. Nearly all proofs of unboundedness use this fact.

3. In our free time ask me whether your favourite group is bounded.

5 Diffeomorphisms

5.1 Diffeomorphisms from functions and vector fields

Let M be a surface with an area form area $\in \Omega^2(M)$ and let $F\colon M \to \mathbf{R}$ be a smooth function. Let $X\colon M \to TM$ be a vector field defined by

$$\iota_X \mathrm{area} = dF. \tag{5.1}$$

Let $\{f_t\} \subseteq \operatorname{Diff}_0(M, \mathrm{area})$ be a 1-parameter family of area preserving diffeomorphisms defined by

$$\left.\frac{d}{dt}\right|_{t=0} f_t(x) = X(x) \tag{5.2}$$

and let $f = f_1$.

Exercise 5.2.

1. Show that formula (5.1) defines an isomorphism between vector spaces of vector fields and 1-forms on M.

2. Draw formula (5.2). Notice that the above construction is just solving ordinary differential equations, or applying the basic theorems on the existence and uniqueness of solutions.

3. Show that $\mathcal{L}_X \mathrm{area} = 0$ and deduce that f is indeed a family of area-preserving diffeomorphisms.

4. Show that $\overline{\mathrm{Flux}}\{f_t\}_{t=0}^{1} = 0$.

5. The group $\mathrm{Ham}(M, \mathrm{area}) = \ker \mathrm{Flux}$ is called the group of Hamiltonian diffeomorphisms of (M, area) and it is a simple group provided M is closed. Show that every $g \in \mathrm{Ham}(M, \mathrm{area})$ can be expressed as $g = f_1 \ldots f_n$, where f_i are diffeomorphisms defined by (5.2). A diffeomorphism defined by (5.2) is called ***autonomous***.

6. Generalise the above construction to time-dependent functions $F\colon M \times [0,1] \to \mathbf{R}$.

7. Let $F'\colon \mathbf{R} \to \mathbf{R}$ be a smooth function such that

$$F'(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t & \text{for } \frac{1}{100} < t < \frac{99}{100} \\ 1 & \text{for } t \geq 1. \end{cases}$$

   Let $F\colon \mathbf{R} \times \mathbf{S}^1 \to \mathbf{R}$ be given by $F(t,z) = F'(t)$. Show that the diffeomorphism $f \in \mathrm{Diff}_0(\mathbf{R} \times \mathbf{S}^1, dt \wedge dz)$ given by (5.2) is a Dehn twist. Notice that by choosing a diffeomorphism $[-1,2] \times \mathbf{S}^1$ with an annulus on any surface we can transplant this Dehn twist to that surface by extending it by the identity away the annulus.

8. Let $\gamma\colon [0,1] \to M$ be an embedded path. Construct an isotopy $\{f_t\}$ such that $f_0 = \mathrm{Id}$, $f_t(\gamma(0)) = \gamma(t)$. Generalise your solution to an immersed path.

9. Let $\mathrm{Mod}(M, (z_1, \ldots, z_n))$ denote the mapping class group of M consisting of mapping classes represented by diffeomorphisms preserving the points $z_1, \ldots, z_n$; the equivalence relation is given by being isotopic through an isotopy preserving z_i's. Show that there is an exact sequence:

$$1 \to \mathbf{P}_n(M) \to \mathrm{Mod}(M, (z_1, \ldots, z_n)) \to \mathrm{Mod}(M) \to 1$$

   It is called the ***Birman exact sequence***.

10. Construct a version of the Birman sequence for the surface braid group $\mathbf{B}_n(M)$.

The idea of a counting quasimorphism has been generalised to hyperbolic groups and to mapping class groups by Bestvina, Bromberg, Calegari and Fujiwara [1, 11]. The following result is a consequence of their ideas and the Birman sequences for both $\mathbf{B}_n(M)$ and $\mathbf{P}_n(M)$.

Theorem 5.3. *The space of homogeneous quasi-morphisms on both $\mathbf{B}_n(M)$ and $\mathbf{P}_n(M)$ is infinite-dimensional. Moreover, for any non-trivial $\gamma \in \mathbf{P}_n(M)$ there exists a homogeneous quasi-morphism $\psi\colon \mathbf{P}_n \to \mathbf{R}$ such that $\psi(\gamma) > 0$.*

Exercise 5.4.

1. Suppose that an element $g \in G$ is conjugate to its inverse. Show that every homogeneous quasi-morphism $\psi\colon G \to \mathbf{R}$ vanishes on g.

2. Give examples of elements g conjugate to their inverses in the following groups: $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$, $\mathrm{SL}(2,\mathbf{Z})$, $\mathbf{B}_n(M)$, where M is a surface and $n \geq 3$.

3. Show that every symmetric matrix $g \in \mathrm{SL}(2,\mathbf{Z})$ is conjugate to its inverse.

5.5 Meaningful generating sets for groups of diffeomorphisms

Exercise 5.6. Let M be a closed manifold. Let G be either the group $\mathrm{Diff}_0(M)$ of all diffeomorphisms of M isotopic to the identity, or $\ker \mathrm{Flux} \subseteq \mathrm{Diff}_0(M,\mathrm{vol})$ if M is equipped with a volume form. In particular, G is a simple group.

1. Show that if $S \subseteq G$ is non-empty and invariant under conjugations then it generates G. This is true for every simple group.

2. Show that the following sets generate G:

   (a) The set of all diffeomorphisms with supports of fixed volume and diffeomorphic to a ball.

   (b) The set of all diffeomorphisms obtained from 1-parameter subgroups generated by vector fields on M.

   (c) The set of autonomous diffeomorphisms.

   (d) The set of diffeomorphism of topological entropy equal to zero.

   (e) Create your own and justify why it is good.

5.7 The Gambaudo-Ghys quasi-morphism

Recall the formula we started with:

$$\boxed{f \mapsto \int_{\mathrm{C}_n(M)} \psi(\gamma(f,x))dx}$$

Exercise 5.8. Suppose that $\psi\colon \mathbf{P}_n(M) \to \mathbf{R}$ be a homogeneous quasi-morphism and that the integral is well defined. Show that the above formula defines a quasimorphism $\Psi\colon \mathrm{Diff}_0(M,\mathrm{vol}) \to \mathbf{R}$.

Exercise 5.9. Prove that the integral above is well defined. The proof in literature is incomplete. It is proven in [2, Lemma 4.1] for the case of the disc, for a specific quasi-morphism in [13, Lemma 4.1], and for $n = 1$ in [17]. Write a bullet-proof argument in full generality.

Let $\Psi\colon \mathrm{Diff}_0(M, \mathrm{vol}) \to \mathbf{R}$ be the homogenisation of the above quasi-morphism:

$$\Psi(f) = \lim_{k\to\infty} \frac{1}{k} \int_{\mathrm{C}_n(M)} \psi(\gamma(f^k, x))dx.$$

Exercise 5.10. Let M be a closed hyperbolic manifold. Prove that Ψ is non-trivial.

1. More precisely, find a quasi-morphism $\psi\colon \pi_1(M) \to \mathbf{R}$ for which Ψ is non-trivial.
2. Show that it is non-trivial on $\ker \mathrm{Flux}$ if M is a surface of genus at least 2.
3. Show that it is non-trivial on $\ker \mathrm{Flux}$ in general.
4. Deduce that the word metric on $\ker \mathrm{Flux}$ associated with a single non-trivial conjugacy class has infinite diameter. In other words, for your favourite diffeomorphism g and a natural number $N \in \mathbf{N}$ there is a diffeomorphism f such that f cannot be expressed as a product of less than N conjugates of $g^{\pm 1}$.

We have thus proven the following result.

Theorem 5.11.
`Insert a good hypothesis here.`
Then the homogeneous quasi-morphism $\Psi\colon \mathrm{Diff}_0(M, \mathrm{vol}) \to \mathbf{R}$ *is non-trivial.*
`Strengthen the statement.`

Exercise 5.12. The aim of this exercise is to show that the diameter of any conjugation-invariant norm on $\mathrm{Diff}_0(\mathbf{S}^n)$ is finite.

1. Let $B \subseteq \mathbf{R}^n$ be a ball of radius 1 centered at the origin. Show that there exists $\theta \in \mathrm{Diff}^c_0(\mathbf{R}^n)$ such that $\theta^i(B) \cap \theta^j(B) = \emptyset$ for all $i \neq j \in \mathbf{N}$.
2. Take for granted that $\mathrm{Diff}^c_0(\mathbf{R}^n)$ is perfect; see [15] for a digested proof. Let $\nu\colon \mathrm{Diff}^c_0(\mathbf{R}^n) \to \mathbf{R}$ be any conjugation-invariant norm. For any $g \in \mathrm{Diff}^c_0(\mathbf{R}^n)$ show that
$$\nu(f) \leq 14\nu(\theta),$$
where θ is a diffeomorphism from the previous item.
3. Let $f \in \mathrm{Diff}_0(\mathbf{S}^n)$. Show that $f = gh$, where $g, h \in \mathrm{Diff}_0(\mathbf{S}^n)$ are such that $\mathrm{supp}(g) \subseteq \mathbf{S}^n \setminus \{x\}$ and $\mathrm{supp}(h) \subseteq \mathbf{S}^n \setminus \{y\}$ and $x \neq y$.
4. Finish the proof.

More generally, according to Tsuboi [18], if M is a closed manifold of dimension different from 2 and 4 then the group $\mathrm{Diff}_0(M)$ is bounded. His proof is a masterclass in differential topology.

Notice that our Theorem 5.11 relies on the fact that either the fundamental group of M admits non-trivial quasimorphism, or that the braid group $\mathbf{P}_n(M)$ does. The theorem gives us nothing if, say, M is simply connected and of dimension at least 3. For example, we have the following.

Open Problem 5.13. *Let* $n \geq 3$. *Is the group* $\mathrm{Diff}_0(\mathbf{S}^n, \mathrm{vol})$ *bounded?*

6 Further reading

In a series of papers, M. Brandenbursky and his collaborators used the Gambaudo-Ghys construction to prove that word metrics on groups of diffeomorphisms associated with meaningful generating sets have infinite diameters (plus some interesting exceptions). These results amount to finding a quasi-morphism $\psi\colon \mathbf{P}_n(M) \to \mathbf{R}$ such that the quasi-morphism Ψ is bounded on a given generating set. The proofs go beyond the scope of this school. Here is a selection of results relevant to this course:

1. The autonomous metric on $\mathrm{Ham}(\mathbf{D}^2, \mathrm{area})$ has infinite diameter [4].
2. Let Σ be a closed oriented surface of genus at least 2. The autonomous metric on $\mathrm{Ham}(\mathbf{\Sigma}, \mathrm{area})$ has infinite diameter [3].
3. The autonomous metric on $\mathrm{Ham}(\mathbf{T}^2, \mathrm{area})$ has infinite diameter [8].
4. The autonomous metric on $\mathrm{Ham}(\mathbf{R}^{2n}, \omega_0)$ has diameter at most 3. It is an open problem whether it is 2 or 3 [6].
5. The entropy metric on $\mathrm{Diff}_0(\Sigma, \mathrm{area})$ has infinite diameter [9]
6. Let M be a complete Riemannian manifold (not necessarily compact). The fragmentation metric on $\mathrm{Diff}_0(M, \mathrm{vol})$ has infinite diameter, provided $\pi_1(M)$ admits non-trivial homogeneous quasi-morphisms [7].
7. The group $[\mathbf{B}_\infty, \mathbf{B}_\infty]$ is unbounded [5].

6.1 What is going on in our course, abstractly

Dynamically. Let G be a group acting on a measure space M, and let H be a group. Let $\gamma\colon G \times M \to H$ be a cocycle and $\psi\colon H \to \mathbf{R}$ a function. We consider the composition $\psi \circ \gamma$ and average in over M to obtain a function on G. Both the cocycle property of γ and ψ being a quasi-morphism ensure that the function has good properties. Measurable cocycles are a basic tool in dynamics and ergodic theory. If you are interested in this direction, read Zimmer's textbook [19].

Exercise 6.2. Let G acts smoothly on an n-dimensional manifold M with trivial tangent bundle. Let $\alpha\colon G \times M \to \mathrm{GL}(n, \mathbf{R})$ be defined by

$$\alpha(g, x) = (dg)^{-1}_{g(x)}\colon T_{g(x)}M = \mathbf{R}^n \to T_x M = \mathbf{R}^n.$$

Show that α is a cocycle. Show that the tangent bundle of any manifold is *measurably* trivial.

Topologically (incorrect version). Let G be a connected topological group acting on a manifold M. Let $p \in M$ and let G_p denotes the stabiliser of p. The group G_p does not have to be connected and we have the quotient

$$G_p \to G_p/(G_p)_0 = \pi_0(G_p).$$

Consider the evaluation fibration:

$$G_p \to G \xrightarrow{\mathrm{ev}_p} M,$$

given by $\mathrm{ev}_p(f) = f(p)$. Assume for simplicity that $\pi_1(M,p)$ has trivial centre. Consider the associated long exact sequence of homotopy groups:

$$\cdots \to \pi_1(G) \xrightarrow{\text{trivial}} \pi_1(M,p) \to \pi_0(G_p) \to 1.$$

It gives an isomorphism $\pi_1(M,p) \to \pi_0(G_p)$. We thus get a surjective homomorphism $G_p \to \pi_1(M,p)$. Composing it with a quasi-morphism $\psi\colon \pi_1(M,p) \to \mathbf{R}$ yields a quasi-morphism on G_p. We would like to extend it to the whole group G. I view our construction as an attempt to do that, although what we constructed is not an extension of a quasi-morphism I described here.

Topologically (correct version). As above, we look at the evaluation fibration

$$G_p \to G \to M.$$

Let $s\colon M \to G$ be a section which is continuous on a set of full measure. Let $\gamma\colon G \times M \to \pi_1(M,p)$ be defined by

$$\gamma(g,x) = s(g(x))^{-1} \circ g \circ s(x).$$

Exercise 6.3. Show that such a section exists. Construct it explicitly. Show that γ is a cocycle.

This is the approach taken by Brandenbursky and Marcinkowski in [10] where they generalised the Gambaudo-Ghys construction to bounded cohomology in all degrees (quasi-morphisms give bounded classes in degree 2).

A Summary of lectures and extra classes

The audience is a mixture of postgraduate and undergraduate students and I slightly rearranged my lectures so that they are useful for less experienced students. Consequently, I haven't presented as much as I planned. For example, I did not discuss the flux homomorphism.

A.1

I defined and gave examples of the following.

- Examples of manifolds (no definition was given).
  - The closed unit disk $\mathbf{D}^2 = \{x \in \mathbf{R}^2 \mid \|x\| \leq 1\}$. The whole course can be given just for the disc for which the main construction and some of the theorems are non-trivial. This is the main example for students who did not attend a course on manifolds.
  - A closed oriented surface Σ_g of genus $g \in \mathbf{N}$. This is also a useful example for students without experience with general manifolds.
  - The n-dimensional sphere $\mathbf{S}^n = \{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}$. I include this example also for less experienced student in order to state an open problem on boundedness of $\mathrm{Diff}_0(\mathbf{S}^n, \mathrm{vol})$.
- I defined a diffeomorphism of a manifold as an infinitely differentiable bijection whose inverse is also infinitely differentiable.
- I explained that the set of diffeomorphisms $\mathrm{Diff}(M)$ is a group.
- We discussed examples of diffeomorphisms of the disc and of surfaces.
  - Rotations of the disc.
  - Pseudo-rotations of the disc.
  - Diffeomorphisms arising from one-parameter subgroups $\mathbf{R} \to \mathrm{Diff}(M)$ obtained from solving ordinary differential equations.
  - I explained how to obtain a vector field form a function: gradient and symplectic gradient. I explained that the symplectic gradient is more interesting for our purposes.
  - I defined a Dehn twist with the use of symplectic gradient.
  - I explained how to *transplant* easy diffeomorphisms from disc in annuli to a surface. I explained that this way we can obtain genuinely complicated diffeomorphisms of surfaces and hence the group $\mathrm{Diff}(M)$ is interesting and mysterious.
- I introduced the compact-open topology on $\mathrm{Diff}(M)$ with the help of a metric.
- In the afternoon I extended my definition to the space $\mathrm{Map}(X, Y)$ of all continuous maps from X to Y.

- I observed that
$$\mathrm{Map}([0,1]\times X, Y) \cong \mathrm{Map}([0,1], \mathrm{Map}(X,Y)).$$
However, in order to define the compact-open topology on the right-hand side space we need to do it properly.
- I introduced the above homeomorphism in order to explain that homotopic maps lie in the same path-connected component of the mapping space.
- I explained the extension
$$\mathrm{Diff}_0(M) \to \mathrm{Diff}(M) \to \pi_0(\mathrm{Diff}(M))$$
and explained that Sergiy will study the mapping class group and I will study the connected component of the identity.
- The rest of the afternoon was a discussion of questions we ask about a (general) group if we want to *understand* it.
- We discussed in a reasonable detail the group $\mathrm{Diff}([0,1])$.
- We shown that the mapping class group is isomorphic to $\mathbf{Z}/2\mathbf{Z}$.
- We shown that there is a surjective homomorphism $\mathrm{Diff}_0([0,1]) \to \mathbf{R}^2$ given by
$$f \mapsto (\log f'(0), \log f'(1)).$$
- But there is more, which we did not discuss. Consider the following equivalence relation on $\mathrm{Diff}_0([0,1])$. Two diffeomorphisms f and g are L-equivalent if there is a $\epsilon > 0$ such that f is equal to g on the interval $[0,\epsilon) \subseteq [0,1]$. Similarly, there an R-equivalence with the interval $(1-\epsilon, 1]$.

  Let G_L denote the group of equivalence classes (show that it is indeed a group). Similarly G_R. Show that there is a surjective homomorphism $\mathrm{Diff}_0([0,1]) \to G_L \times G_R$.
- It is a slightly less trivial fact that the groups G_L and G_R are simple and that the kernel K of the above homomorphism is also simple. In other words, the extention
$$K \to \mathrm{Diff}_0([0,1]) \to G_L \times G_R,$$
cannot be further decomposed.

A.2

- I discussed transitivity properties of diffeomorphisms, which led to a definition of configuration spaces and braid groups.
- I mentioned that braid groups are interesting for graphs and surfaces and don't bring anything new beyond the fundamental group for higher dimensional manifolds.
- I gave the presentation of the standard braid group and discussed a surjective homomorphism $\mathbf{B}_n \to \mathbf{Z}$ and gave a number of hints for proving that this is the abelianisation.
- In the extra class I introduced and discussed presentations of groups.

A.3

Stefan stated a theorem that the braid group $\mathbf{B}_n$ is torsion-free. His suggested proof was that it acts on a contractible space. I tried to give an argument in class but is was a bit too involved that I initially thought. Here is a clarification.

Definition A.4. A path-connected topological space is called ***aspherical*** if its universal cover $\widetilde{X}$ is contractible.

For sensible spaces, such as simplicial complexes, asphericity is equivalent to vanishing all higher homotopy groups. That is, a simplicial complex is aspherical if and only if $\pi_k(X) = \{0\}$ for $k \geq 2$.

Exercise A.5. Let $F \to E \to B$ be a fibre bundle, where the spaces are homotopy equivalent to simplicial complexes. Suppose that both F and B are aspherical. Use the long exact sequence of homotopy groups of the fibration to show that E is aspherical.

Theorem A.6. *The configuration space* $\mathrm{C}_n(\mathbf{C})$ *is aspherical.*

Proof. Let $f_n \colon \mathrm{C}_n(\mathbf{C}) \to \mathrm{C}_{n-1}(\mathbf{C})$ be the forget map: $f_n(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. It is a fibration

$$\mathbf{C} \setminus \{x_1, \dots, x_{n-1}\} \to \mathrm{C}_\mathrm{n}(\mathbf{C}) \to \mathrm{C}_{n-1}(\mathbf{C}). \tag{A.1}$$

The proof is by induction. The base of the induction is satisfied, since $\mathrm{C}_1(\mathbf{C}) = \mathbf{C}$ is contractible, hence aspherical. The induction step follows from the exercise above. □

By definition, $\mathrm{S}_n(\mathbf{C}) = \mathrm{C}_n(\mathbf{C})/S_n$, where the action of the symmetric group S_n on the configuration space $\mathrm{C}_n(\mathbf{C})$ is free. It follows that the spaces $\mathrm{S}_n(\mathbf{C})$ and $\mathrm{C}_n(\mathbf{C})$ have the same universal cover, which by the above theorem is contractible. Consequently, $\mathbf{B}_n = \pi_1(\mathrm{S}_n(\mathbf{C}))$ acts freely on its universal cover. Moreover, the universal cover is finite dimensional (it is of dimension $2n$).

Stefan uses the following fact to finish the argument. Unfortunately, its proof requires a little bit more machinery.

Theorem A.7. *If a group G acts freely and properly discontinuously on a finite dimensional contractible simplical complex then it is torsion-free.*

However, we can still prove that the pure braid groups are torsion-free.

Corollary A.8. *The pure braid groups $\mathbf{P}_n$ for $n \geq 2$ is torsion-free.*

Proof. Since the configuration spaces $\mathrm{C}_n(\mathbf{C})$ are aspherical, the long exact sequence of homotopy groups of the fibration (A.1) yields an extension

$$\mathbf{F}_{n-1} \to \mathbf{P}_n \to \mathbf{P}_{n-1}.$$

We know that $\mathbf{P}_2 \cong \mathbf{Z}$ is torsion free and we know that free groups are torsion free. Assume, by induction that $\mathbf{P}_{n-1}$ is torsion free. Let $g \in \mathbf{P}_n$. If $g \in \mathbf{F}_{n-1}$ then it is of infinite order. If, on the other hand, it projects to a non-trivial element in $\mathbf{P}_{n-1}$ then it is of infinite order, by induction hypothesis. □

A.9

I discussed commutator subgroups of groups.

- I showed that $[\mathbf{F}_2, \mathbf{F}_2] \cong \mathbf{F}_\infty$ to indicate that commutator subgroups of *easy* groups are *complicated.*

- I proved that the commutator subgroup $[\mathbf{B}_n, \mathbf{B}_n]$ is perfect for $n \geq 5$ by solving exercises from Section 2.

- I introduced general Artin groups and indicated that similar arguments work for some of them.

- In the extra time I proved that the fundamental group of the complement of the trefoil knot is isomorphic to the braid group $\mathbf{B}_3$.

- I proved that the pure braid group $\mathbf{P}_3$ isomorphic to $\mathbf{Z} \times \mathbf{F}_2$. I made a mistake in class (and claimed a weaker statement). Here is the correct argument. Consider the extension
$$\mathbf{F}_2 \to \mathbf{P}_3 \to \mathbf{P}_2 = \mathbf{Z}.$$
Observe that the central element Δ^2 maps to a generator of $\mathbf{P}_2$ (draw a picture!). Construct a section taking this generator to Δ^2 and argue as before that the extension has to be the product.

A.10

- I explained the construction of the Gambaudo-Ghys cocycle $\gamma\colon \mathrm{Diff}_0(M) \times \mathrm{C}_n(M) \to \mathbf{P}_n(M)$ and discussed all its problems. Students suggested that it is possible to make the choice of the paths ℓ_x so that the cocycle γ is defined for all x. Their argument was to use the fact that the configuration space $\mathrm{C}_n(M)$ is path-connected.

- I defined quasi-morphisms and gave basic examples and proved that the formula (1.1) yields a quasi-morphism on the group of *volume preserving* diffeomorphisms.

- I briefly mentioned applications.

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