GROUP ACTIONS IN SYMPLECTIC GEOMETRY

JAREK KĘDRA

Let (M, ω) be a closed symplectic manifold. The group of all symplectic diffeomorphisms of (M, ω) is denoted by $\mathrm{Symp}(M, \omega)$. It is the group of all diffeomorphisms preserving the symplectic form. There is a connected subgroup $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$, called the group of Hamiltonian diffeomorphisms. It consists of those diffeomorphisms generated by the flows of vector fields induced by functions. Such a time dependent vector field X_t is defined by

$$i_{X_t}\omega = dH_t,$$

where $H_t\colon M \to \mathbb{R}$. An action $G \to \mathrm{Ham}(M, \omega)$ is called Hamiltonian.

Question. *What groups G admit Hamitlonian actions on closed symplectic manifolds?*

There are many nontrivial examples of Hamiltonian actions of compact groups. They are very well understood [5]. On the other hand, there is the following elementary result by Delzant [1].

Theorem (Delzant). *If a connected semisimple Lie group G admits a Hamiltonian action on a closed symplectic manifold (M, ω) then it is compact.*

This motivates the question whether there are interesting Hamiltonian actions of infinite discrete groups like, for example, lattices in semisimple Lie groups. In turns out that, under certain geometric conditions, there are restrictions.

Let $p\colon \widetilde{M} \to M$ be the universal cover. A symplectic form ω on M is called *hyperbolic* if its pull back $p^*\omega$ admits a primitive $\alpha \in \Omega^1(\widetilde{M})$ bounded with respect to the Riemannian metric induced from a Riemannian metric on M. A manifold equipped with a hyperbolic symplectic form is called *symplectically hyperbolic.*

Examples of symplectically hyperbolic manifolds:

(1) A surface of genus at least two; a product of such surfaces;
(2) A symplectic manifold admitting a Riemannian metric of negative sectional curvature;

(3) A symplectically aspherical manifold with hyperbolic fundamental group; recall that a symplectic manifold M is called *symplectically aspherical* if every two-sphere in M has trivial symplectic area;
(4) A symplectic manifold such that the cohomology class of the symplectic form is bounded, that it is represented by a bounded singular cochain.

Theorem (Polterovich [6]). *Let (M, ω) be a closed symplectically hyperbolic manifold. If $\Gamma \subset \mathrm{Ham}(M, \omega)$ is a finitely generated subgroup then all cyclic subgroups of Γ are undistorted.*

A cyclic subgroup G of a finitely generated group Γ is called *undistorted* if there exists a positive constant $C > 0$ such that

$$|g^n| \geq C \cdot n,$$

where $g \in G$ is a generator. The norm $|g|$ is the word norm with respect to the finite set of generators.

Examples of groups with undistorted cyclic subgroups:

(1) a free nonabelian cyclic subgroup;
(2) a right angled Artin group;
(3) a hyperbolic group;

Examples of distorted cyclic subgroups:

(1) a finite cyclic subgroup;
(2) the centre of the uppertriangular 3×3-matrices with integer entries;
(3) a non-uniform irreducible lattice in a semisimple Lie group of higher rank admits distorted cyclic subgroups; for example $\mathrm{SL}(n, \mathbb{Z})$, for $n > 2$ [4].

It follows from the Polterovich theorem that irreducible lattices in semisimple Lie groups of higher rank do not admit effective Hamiltonian actions on symplectically hyperbolic manifolds.

An alternative proof of the Polterovich theorem has been given by Gal and Kędra in [2]. The original proof by Polterovich is published in [6]. Examples of hyperbolic symplectic forms are discussed by Kędra in [3].

References

[1] Thomas Delzant. Sous-algebres de dimension finie de l'algebre des champs hmiltoniens. *unpublished*.

[2] Światosław Gal and Jarek Kędra. A cocycle on the group of symplectic diffeomorphisms. *Advances in Geometry*, to appear.

[3] Jarek Kędra. Symplectically hyperbolic manifolds. *Differential Geom. Appl.*, 27(4):455–463, 2009.

[4] Alexander Lubotzky, Shahar Mozes, and M. S. Raghunathan. The word and Riemannian metrics on lattices of semisimple groups. *Inst. Hautes Études Sci. Publ. Math.*, (91):5–53 (2001), 2000.

[5] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[6] Leonid Polterovich. Growth of maps, distortion in groups and symplectic geometry. *Invent. Math.*, 150(3):655–686, 2002.

University of Aberdeen and University of Szczecin
E-mail address: kedra@abdn.ac.uk