Lagrangian submanifolds of the standard $\mathbf{C}^n$

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In this talk I will present some restrictions on the topology of a monotone odd dimensional Lagrangian submanifold of the standard symplectic Euclidean space.

$$L \hookrightarrow \mathbf{C}^n$$

Essentially, a monotone lagrangian in $\mathbf{C}^n$ cannot have too complicated topology. For example, it can't admit a metric of negative curvature, in dimension three it has to be a product $\mathbf{S}^1 \times \Sigma$ and, in general, its simplicial volume has to be zero.

Plan of the talk

- Symplectic vector space and the lagrangian grassmannian
- Symplectic manifolds and Lagrangian immersions
- Lagrangian submanifolds in $\mathbf{C}^n$
- The main result and its consequences
- Comments about the proof
- Examples of monotone embeddings.

Symplectic vector space and the lagrangian grassmannian

- (V, ω) - symplectic vector space: V is a real $2n$-dimensional vector space and ω is a nondegenerate skew-symmetric bilinear form.
- For example, $\mathbf{C}^n$ and $\omega = \sum dx^i \wedge dy^i$.
- Every symplectic vector space is isomorphic to the above example.
- A subspace $L \subset \mathbf{C}^n$ is called *Lagrangian* if the symplectic form ω vanishes on L and $\dim L = n$.
- For example, every real line on the plane $\mathbf{C}$ is Lagrangian.

- The space of all Lagrangian subspaces in $\mathbf{C}^n$ is called *Lagrangian grassmannian* and it is denoted by $\Lambda(n)$.
- Exercise: $\Lambda_n = U(n)/O(n)$.
- Consequently:

$$\pi_1(\Lambda_n) = H_1(\Lambda_n; \mathbf{Z}) = H^1(\Lambda_n; \mathbf{Z}) = \mathbf{Z}.$$

- The square of the determinant:

$$\det^2\colon \Lambda_n \to \mathbf{S}^1 \subset \mathbf{C}^\times$$

defines a bundle $SU(n)/SO(n) \to \Lambda_n \to \mathbf{S}^1$ and the pullback of the length form on the circle represents a generator $\mu \in H^1(\Lambda_n; \mathbf{Z})$ called the *universal Maslov class*.

Symplectic manifolds and Lagrangian submanifolds

- (M, ω) is called a symplectic manifold if M is a smooth manifols and ω is nondegenerate and closed two-form.
- (T_pM, ω_p) is a symplectic vector space.
- If $\omega = d\alpha$ then $(M, d\alpha)$ is called *exact*.
- For example, $(\mathbf{C}^n, dx^i \wedge dy^i = d(x^i dy^i))$ is exact.
- An immersion $f\colon L \to M$ is called Lagrangian if $f^*\omega = 0$ and $\dim L = \frac{1}{2}\dim M$. In such a case, $df(T_xL) \subset T_{f(x)}M$ is a Lagrangian subspace is the symplectic vector space.
- For example, any one-dimensional immersion is Lagrangian in a symplectic surface.
- The standard sphere $\mathbf{S}^2 \subset \mathbf{R}^3 \subset \mathbf{C}^2$ is not a Lagrangian submanifold.

Lagrangians in $\mathbf{C}^n$

$(\mathbf{C}^n, \omega = \sum dx^i \wedge dy^i = d\alpha)$ - the standard symplectic space

- $f : L \to \mathbf{C}^n$ a Lagrangian immersion.
- $0 = f^*\omega = f^*d\alpha = df^*\alpha$ $\qquad [f^*\alpha] \in H^1(L; \mathbf{R})$.
- $G_f : L \to \Lambda_n$ – the Gauss map $\qquad \mu_f := f^*\mu \in H^1(L; \mathbf{Z})$.

Monotonicity:

$$[f^*\alpha] = K \cdot \mu_f$$

for some $K > 0$.

If $K = 0$ then L is called *exact*.

Gromov proved that there are no exact Lagrangian embeddings into $\mathbf{C}^n$. In particular, there are no Lagrangian embeddings of simply connected manifolds.

Theorem [Evans–K.]

Let L be a closed oriented spin odd-dimensional manifold which is a connected sum of aspherical manifolds. If $f\colon L \to \mathbf{C}^n$ is a monotone Lagrangian embedding then there exists a smooth map

$$\mathbf{S}^1 \times M \to L$$

of nonzero degree, where M is an oriented closed manifold.

Corollary

- There exists a surjection $\mathbf{Z} \times \Gamma \to \pi_1(L)$; in particular $\pi_1(L)$ has infinite centre.
- $\pi_1(L)$ is not hyperbolic.
- L does not admit a Riemannian metric of negative curvature.
  Remark: Eliashberg and Viterbo obtained the last statement without the monotonicity assumption.

Corollary

- Every summand of L has vanishing simplicial volume.
- If $\dim L = 3$ then $L = \mathbf{S}^1 \times \Sigma$. Remark: Fukaya obtained the same statement without the monotonicity assumption but assuming that L is prime.
- If $f\colon L \to \mathbf{C}^3$ is a Lagrangian immersion with k double points then resolving the double points can produce a monotone embedding only if $L = \mathbf{S}^3$ and $k = 1$.

We have a monotone Lagrangian embedding:

$$f\colon L \to \mathbf{C}^n.$$

Let $\mathbf{M}_{0,1}(a, J)$ denote the moduli space of J-holomorphic discs in $\mathbf{C}^n$ with the boundary on L and with one marked boundary point such that the boundary represents a free homotopy class a. There exists a free homotopy class a of loops in L such that the evaluation map

$$\mathrm{ev}\colon \mathbf{M}_{0,1}(a, J) \to L$$

has nonzero degree. To obtain this statement we use a result of Damian [Commentari Math. Helv. 87] which implies that for any J and for any $x \in L$ there exists a J-holomorphic dics $u\colon \mathbf{D}^2 \to \mathbf{C}^n$ with boundary on L passing through x such that

$$\langle \mu_f, u(\partial \mathbf{D}^2) \rangle = 2.$$

The strategy of the proof

The moduli space $\mathbf{M}_{0,1}(a, J)$ admits a free circle action (rotate the source) with the quotient $\mathbf{M}_{0,0}(a, J)$ – the moduli space of J-holomorphic discs representing a. There exists a finite cover $\overline{\mathbf{M}_{0,0}(a, J)} \to \mathbf{M}_{0,0}(a, J)$ such that

$$\begin{array}{ccccc} \overline{\mathbf{M}_{0,0}(a, J)} \times \mathbf{S}^1 & \longrightarrow & \mathbf{M}_{0,1}(a, J) & \xrightarrow{\text{ev}} & L \\ \downarrow & & \downarrow & & \\ \overline{\mathbf{M}_{0,0}(a, J)} & \longrightarrow & \mathbf{M}_{0,0}(a, J) & & \end{array}$$

The composition of the maps in the top row gives the required map of nonzero degree. $\square$

Examples of monotone Lagrangian embeddings

Monotone Lagrangian immersions obey the *h*-principle.

Theorem

If $f\colon L \to \mathbf{C}^n$ is a K-monotone Lagrangian immersion and $e\colon K \to \mathbf{C}^m$ be a K-monotone Lagrangian embedding. Then there is a monotone Lagrangian embedding

$$K \times L \to \mathbf{C}^{m+n}.$$

Example

- Let Σ be a closed oriented surface. It admits a monotone Lagrangian immersion into $\mathbf{C}^2$ and hence $\Sigma \times \mathbf{S}^1$ admits a monotone Lagrangian embedding into $\mathbf{C}^3$.
- If M is a closed oriented three manifold then $M \times \mathbf{S}^1$ admits a monotone Lagrangian embedding into $\mathbf{C}^4$.

Thank you for listening!